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# Stochastic Modelling of Chaotic Time Series

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**Abstract**—Recently methods have been developed that model low–dimensional chaotic systems in terms of stochastic differential equations. We test such methods in an electronic circuit experiment. We achieved to obtain reliable drift and diffusion coefficients even without a pronounced time scale separation of the chaotic dynamics. By comparing analytical solutions of the corresponding Fokker–Planck equation with experimental data we show that crisis induced intermittency can be described in terms of a stochastic model which is dominated by state space dependent diffusion.

## 1. Introduction

Since the pioneer works on Brownian motion by A. Einstein[1] and P. Langevin[2] stochastic modelling has become a well established method. It is a textbook example that one can model a system with a distinct time scale separation between some slow and many fast degrees of freedom by a stochastic differential equation. Such an equation describes the evolution of the slow degrees of freedom while the fast ones are replaced by some heat bath that leads to suitable stochastic forces.

Here we report on our investigation on stochastic modelling of an experimental system[3]. Our investigation is motivated by the quite recent finding that stochastic modelling is not only applicable in systems with many fast degrees of freedom but also in low–dimensional chaotic systems[4]. Our experimental system has only three degrees of freedom and shows crisis–induced intermittency. Intermittency is always a phenomenon with two timescales – a fast oscillation on the pre–critical attractors and a slow switching between them. Therefore it is a good candidate for stochastic modelling. Since the found time scale separation is not pronounced, it consequently is not obvious that stochastic modelling works. Therefore our investigation might be interpreted as a test of the extreme limits of stochastic modelling.

The paper is organised as follows: In the following section we give details on the experimental system and the relevant control parameter range where intermittency is observed. Section 3 describes the time series analysis that allowed us to model the dynamics in terms of a Fokker–Planck equation. A detailed analysis of the Fokker–Planck equation and comparison between the theoretical and experimental results can be found in section 4.

## 2. Experimental System

The time series analysis is based on data from a Shinriki oscillator (see fig.1(a))[5]. This circuit belongs to the Chua family of circuits[6]. The autonomous circuit consists of a RLC–element coupled through a non–linearity with an additional $RC$–element. The essential non–linearity consists of a pair of Zener diodes $Z_{1,2}$ (BZX85C3V3). There current–voltage characteristic (cf. inset in fig.1(a)) can be described by: $I_D(V_1 - V_2) = I_D(\Delta V) = \mathrm{sgn}(\Delta V) f(|\Delta V| - V_Z)$ where $V_Z = (1.02 \pm 0.04)\mathrm{V}$ and $f(x) = (Ax^2 + Bx^3)\Theta(x)$ denotes a third order fit with parameter values $A = (13.1 \pm 0.7)\mathrm{mA/V}^2$ and $B = (-1.59 \pm 0.15)\mathrm{mA/V}^3$. In parallel to the control parameter $R$ there is an operational amplifier circuit (AD711JN) which acts as a negative resistor with constant resistance $-R_N$ and provides a power supply for the circuit.

To measure the voltage $V_1$ we used a transient recorder card (Meilhaus ME2600). The digital output channels of this cards were used for online variation of the control parameter $R$, which was realized by several digital resistors (Xicor X9C102/4P). The fully computer controlled experiment enabled us to measure long time series of the circuit for many different control parameter values.

The dynamics of the circuit is shown in the bifurcation diagram in fig.1(b). The data is typical for a Chua–type circuit: For increasing $R$ a period doubling route to chaos is found. Several periodic windows occur for $R \geq 53\mathrm{k}\Omega$ before at $R = R_c \approx 66\mathrm{k}\Omega$ the mono–scroll attractor collides with its counterpart and a double–scroll attractor is observed. For $R \geq R_c$ the circuit shows crisis–induced intermittency. Our time series analysis focuses on this intermittency regime where the dynamics shows two time scales: A fast oscillation on the chaotic saddles (the former stable mono–scroll attractors) and a slow jumping dynamics between them (cf. fig.2). Details on the crisis–induced intermittency can be found elsewhere[7]. Here we just

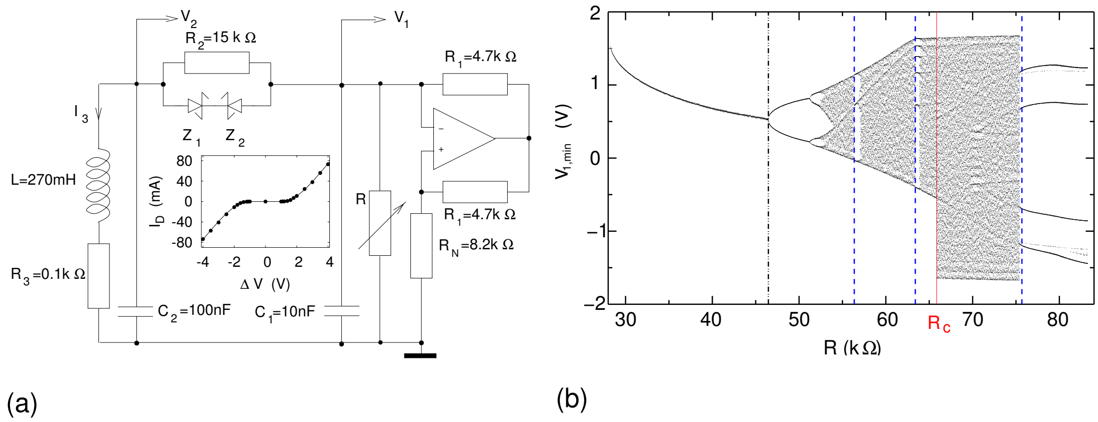


Figure 1: (a) Schematic of the Shinriki oscillator. Inset: experimental data (symbols) and analytic data fit of the current–voltage characteristic of the Zener diodes $Z_{1,2}$. (b) Measured bifurcation diagram for a initial voltage $V_1 > 0$. The minimum voltage $V_1$ is shown as a function of the control parameter R.

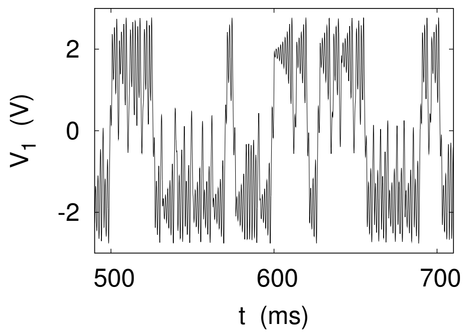


Figure 2: Sample of $V_1(t)$ time series for $\Delta R \approx 1.1\text{k}\Omega$ measured with a sampling time of $80\mu$s. About 150 oscillations are shown and 10 jumps between the two states occur.

| $\Delta R = R - R_c$ | mean residence time $\tau$ | $\tau/T_{\text{osc}}$ |
|---|---|---|
| $100\Omega$ | 0.1s | $\approx 70$ |
| $600\Omega$ | 30ms | $\approx 20$ |
| $3\text{k}\Omega$ | 3ms | $\approx 2$ |

Table 1: Mean residence time and the ratio between it and the average oscillation period of the system for different values of $\Delta R$.

want to stress that (a) for a wide range of $\Delta R$ the mean residence time obeys a typical scaling law[8] $\tau \sim \Delta R^{-\gamma}$ with $\Delta R = R - R_c$ and $\gamma \approx 0.7$ (b) for typical values of $\Delta R$ there are about 10–100 oscillations on a chaotic saddle before a jump occurs (c.f. table2). In typical textbook examples – e.g. the motion of a particle in a double well driven by a Gaussian noise with intensity $\sigma^2$ – one finds (a) a mean residence time that scales like $\exp(1/\sigma^2)$ and (b) a time scale separation of several orders of magnitude for typical noise intensities. Consequently the dynamics of the circuit does not resemble the typical features for which stochastic modelling is obvious.

## 3. Time series analysis

Assume a non–linear dynamical system $\dot{x} = f(x)$ and a scalar measurement $z(t) = g[x(t)]$. The scalar time series might be e.g. a component of the state vector $x$ or some non–linear function of the components. A stochastic differential equation that will describe the dynamics of $z(t)$ will be a Langevin equation:

$$\dot{z} = h\,(z) + k\,(z)\,\xi(t), \tag{1}$$

where $h(z)$ is a deterministic force and $k\,(z)\,\xi(t)$ is Gaussian white noise with state dependent intensity $k^2$. One way to quantify the stochastic dynamics is to analyse the corresponding Fokker–Planck equation:

$$\dot{\rho}(z,t) = \left[\frac{\partial}{\partial z} D_1(z) + \frac{\partial^2}{\partial z^2} D_2(z)\right] \rho(z,t). \tag{2}$$

$D_1$ is the drift and $D_2$ the diffusion coefficient that govern the evolution equation of the probability density $\rho$. There exists a standard recipe to obtain these coefficients from a stochastic time series by evaluating first and second moments[9]:

$$\begin{aligned} \Delta t\, D_1(Z) &= \langle (z(t+\Delta t) - z(t)) \rangle + o(\Delta t) \quad \text{(3a)} \\ 2\Delta t\, D_2(Z) &= \langle (z(t+\Delta t) - z(t))^2 \rangle + o(\Delta t) \quad \text{(3b)} \end{aligned}$$

where the averages are taken for the ensemble with $z(t) = Z$. It can be shown that for stochastic time series in the asymptotic limit $\Delta t \to 0$ eqs. (3) yields reliable values for $D_1$ and $D_2$.

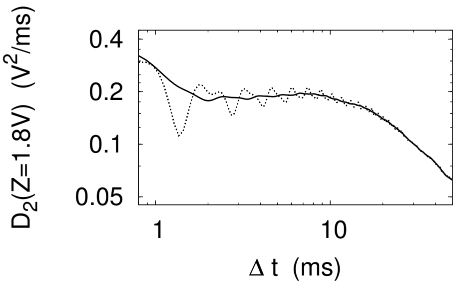


Figure 3: Dependence of the diffusion coefficient $D_2$ on the time delay $\Delta t$ (c.f.eq. (3b)): $D_2$ evaluated at $Z = 1.8\mathrm{V}$ from a time series measured at $\Delta R \approx 1.1\mathrm{k}\Omega$. Raw data (dotted line) and a sliding average over one period $T_{\mathrm{osc}}$ is shown. For $\Delta t \in [2\mathrm{ms}, 8\mathrm{ms}]$ a plateau appears. For large $\Delta t$ values the characteristic power law decay can be seen.

Obviously the limit $\Delta t \to 0$ is not applicable in deterministic systems since on this time scale the dynamics is strongly correlated. A suitable $\Delta t$ value has to be large compared to the correlation time of the fast chaotic motion but apart from this constraint eqs. (3) have to be evaluated in the asymptotic limit of small $\Delta t$. It is of cause not guaranteed that a stochastic model in terms of eq. (2) will capture the essential features of a deterministic time series $z(t)$. Eq. (2) and the time series analysis based on eqs. (3) rely on the possibility to describe $z(t)$ by a Markovian process. Although it is not clear whether our circuit data fulfils this requirement, we will nevertheless first model the data and test for such properties later.

Our experimental data is the voltage $z(t) = V_1(t)$. Each of the analysed time series consisted of $6 \times 10^5$ data points measured with a sampling time of $80\mu\mathrm{s}$. Evaluation of eq. (3) typically results in a dependence between the moment and the time delay $\Delta t$ that can be seen in fig. 3: For intermediate values $\Delta t \in [2\mathrm{ms}, 8\mathrm{ms}]$ a plateau is found. Thus, values in this range can be used to estimate the coefficients of eq. (2).
To estimate spatially resolved $D_1$ and $D_2$ we choose $\Delta t = 4.5\mathrm{ms}$. The conditional averages in eqs. (3) have been computed on a spatial grid with a step size of $\Delta Z = 50\mathrm{mV}$. This grid leads to an ensemble size of approximate $5 \times 10^3$ for each data point. We made sure that the final results are stable against these particular choices.
In fig. 4 $D_1$ and $D_2$ evaluated with $\Delta t = 4.5\mathrm{ms}$ at $\Delta R = 1.1\mathrm{k}\Omega$ are shown. $D_1$ is dominated by a linear decay with a superimposed oscillatory structure. $D_2$ turns out to be unimodal: Large values around $Z = 0\mathrm{V}$ and small values at the boundaries. These large scale features are robust against reasonable changes of the time series analysis parameters $\Delta t$ and $\Delta Z$. The fine structure shows slight changes for varying parameters

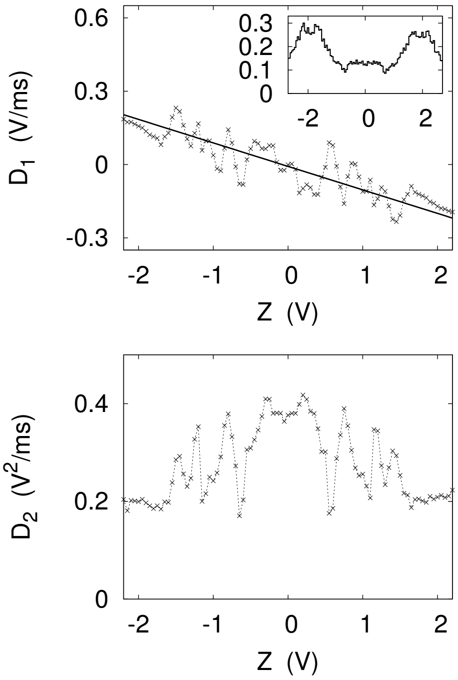


Figure 4: Drift $D_1$ (top) and diffusion $D_2$ (bottom) as a function of $Z = V_1$ evaluated with $\Delta t = 4.5\mathrm{ms}$ for $\Delta R = 1.1\mathrm{k}\Omega$. The straight line indicates a linear least square fit. Inset: Histogram of $V_1$ with resolution $\Delta Z$ indicating the stationary distribution.

of the analysis. A more pronounced change of them can be seen for different lengths of the time series. This indicates that they are partially resulting from finite ensemble sizes. Some peak structures even seem to be an artifact generated by the spiky time series. Cf. the peaks in $D_2$ for $z \in \pm[0.7\mathrm{V}, 1.4\mathrm{V}]$ with the stationary distribution (shown as a inset in fig. 4) where no such feature occurs. The stable large scale results achieved lead to our first conclusion that despite the lack of time scale separation the time series analysis leads to reproducible drift and diffusion coefficients.

## 4. Analysis of the stochastic model

The simplest model describing the data given in fig. 4 is to approximate the drift by a linear expression $D_1(z) = -\alpha z$ and the diffusion by a parabola $D_2(z) = D(z_0^2 - z^2)$. Such a model is quite simplistic since it does not take the oscillating part into account but allows us to solve the corresponding Fokker–Planck equation (2) analytically.
The stationary distribution of eq. (2) is given by $\rho_* \sim (z_0^2 - z^2)^{\alpha/(2D)-1}$. For $\alpha < 2D$ the bimodal distribution reproduces the measured stationary distribution (cf. inset in fig. 4). The eigenvalues of the Fokker–Planck operator governing the time evolution

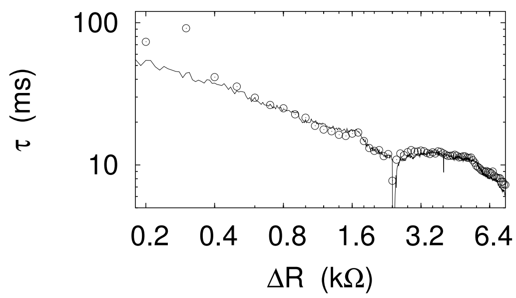


Figure 5: Mean residence time $\tau$ as a function of the control parameter $\Delta R$. The line shows experimental data measured with a resolution of $10\Omega$. The symbols give the analytical determined values of the stochastic model.

read $\Lambda_n = -n\alpha - n(n-1)D$. For $\alpha \ll D$ there appears a spectral gap and the mean residence time of the jumping process between the two states is given by $\tau = 2/\alpha$.

We evaluated the analytical estimate by computing the slope of the linear trend $\alpha$ using a least square fit in the interval $[-1.6\text{V}, 1.6\text{V}]$. This investigation is based on time series from the circuit measured with a control parameter resolution of $100\Omega$. A comparison between these values and direct measured residence times is given in fig. 5. Taking into account the simplicity of the model the agreement between the experimental and theoretical results is quite accurate over a wide control parameter range. We conclude that the stochastic model captures the intermittent features of the slow time scale.

Although the accuracy of the mean residence time seems to confirm the validity of the Fokker–Planck model we checked for higher–order coefficients in the Kramers–Moyal expansion (3). At least the third– and fourth–order terms do not vanish. A strict test for Markovian property of the time series was not possible with the given ensemble size. But using conditional correlation functions we were able to estimate that non–Markovian effects are visible at time scales of about 20ms. Apparently this shortcoming and the simplistic nature of the model, that neglects the fine structure of drift and diffusion, are of minor importance for the modelling of the measured intermittency as demonstrated by the results of the analysis.

## 5. Conclusion

Our investigation shows that even when no excessive time scale separation is present one may successfully model low–dimensional chaotic systems by stochastic forces. We have demonstrated that reliable drift and diffusion coefficients can be evaluated using the Kramers–Moyal expansion with a time delay that has been adapted to the deterministic nature of the data.

Although our simple stochastic model approximates only the large scale features of the drift and diffusion, the essential features of the intermittency are captured. The analytical determined mean residence times are in good agreement with experimental data over a wide control parameter range and so are the analytical and experimental stationary distribution. The model describes crisis induced intermittency as a tunnelling which is dominated by state space dependent diffusion.

The modelling of chaotic motion by suitable stochastic forces could have a wide range of applications even for low–dimensional systems with no pronounced time scale separation. Stochastic models may reveal features about the underlying non–linear dynamics but certainly allow a different point of view. Therefore this modelling is more than just a practical tool that reduces a non–linear system to a one dimensional stochastic model.

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