2015 International Symposium on Nonlinear Theory and its Applications
NOLTA2015, Kowloon, Hong Kong, China, December 1-4, 2015

Uncontrolled manifold analysis of oscillatory motions in dynamical models of body movement based on the Floquet theory

Sho Shirasaka†, Wataru Kurebayashi‡, and Hiroya Nakao†

†Graduate School of Information and Engineering, Tokyo Institute of Technology
O-okayama 2-12-1, Tokyo 152-8552, Japan
‡Faculty of Software and Information Technology, Aomori University
Koubata 2-3-1, Aomori 030-0943, Japan
Email: shirasaka.s.aa@m.titech.ac.jp

Abstract—The uncontrolled manifold analysis provides the synergy index, which quantifies how well a neuro-musculo-skeletal system of the animal body is coordinated to maintain a certain movement such as walking. The synergy index has experimentally been estimated from observed data. In this study, we propose a theory that gives the synergy index of a stable rhythmic movement from dynamical models by using the Floquet theory for limit cycles. The synergy index of an oscillatory motion in a simple model is calculated as an illustration of the theory.

1. Introduction

The body of animals has redundant degrees of freedom (DOFs) to solve motor tasks in their voluntary movements. For example, the arms and legs have more joints than needed to specify the spatial position and orientation of the hands and feet, and the muscles are influenced by more motoneurons than needed to generate identical muscle activities [1]. How animals coordinate the redundant DOFs has been recognized as a central problem in the study of motor control and learning [2].

Uncontrolled manifold (UCM) analysis is a quantitative approach to this problem, which is based on the dynamical systems perspective [3]. The UCM analysis provides the synergy index, which quantifies the extent to which the redundant DOFs is coordinated in stabilizing the performance of motor tasks under internal and external perturbations, and how flexible the coordination is. The synergy index has experimentally been estimated and analyzed for a number of functional tasks by the UCM analysis. However, the relation between the synergy index and the underlying dynamics of the motor behavior remains unexplored theoretically.

If a theory that relates the synergy index to the system parameters such as the geometric parameters of the body and viscoelasticity of actuators is established, it will facilitate parametric sensitivity analysis of the synergy in motor tasks. It would also make optimization and assessment of the robustness of the synergy index more tractable when designing devices such as orthosis, prosthesis, exoskeletal systems and autonomous robots. Furthermore, theoretical knowledge of the dependence of the synergy index on the body properties can be a guiding principle for clarifying the causes of motor disorders and can give some implications for recovery and rehabilitation [4].

In this study, we propose a theory that relates the synergy index of stable periodic motions in dynamical systems subjected to weak white Gaussian noise on the basis of the Floquet theory. The fundamental properties of the Floquet vectors [5] simplify the analysis of the covariance matrix of the periodic motion, from which the synergy index can be calculated. We apply our theory to a simple model of oscillatory motions and compare the results with direct numerical simulation of the model equation.

2. UCM analysis using the Floquet theory

2.1. Model

We consider a randomly perturbed dynamical system for a state variable $\boldsymbol{x} \in \Omega \subset \mathbb{R}^n$,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \epsilon \boldsymbol{P}(\boldsymbol{x})\xi(t), \tag{1}$$

where $\boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^n$ is a vector field describing the unperturbed dynamics, $\boldsymbol{P}(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$ is a matrix function, and $\xi(t) \in \mathbb{R}^n$ is the random perturbation whose elements $\xi_i(t)$ $(i = 1, 2, \cdots, n)$ are mutually independent white Gaussian noise of zero mean and unit variance. The constant $\epsilon \geq 0$ determines the noise intensity and is assumed to be small. We interpret the stochastic differential equation (1) in the Ito sense [6]. It is assumed that the system Eq. (1) has a linearly stable limit-cycle solution $\boldsymbol{x}_0(t)$ with a period T, i.e. $\boldsymbol{x}_0(t) = \boldsymbol{x}_0(t + T)$ when it is unperturbed ($\epsilon = 0$).

2.2. Floquet vectors

The stability of a limit cycle is characterized by the Floquet multipliers, which are the eigenvalues of the monodromy matrix of the limit cycle, and by the associated eigenvectors called the Floquet vectors [5]. We denote the Floquet vectors as $\{\boldsymbol{z}_i(t_*)\}_{i=1}^n$, which are defined at each point on the limit cycle $\boldsymbol{x}_0(t_*)$. If the limit cycle is linearly stable, one of the Floquet multipliers is unity and the other multipliers lie within the unit circle. We denote the Floquet vector with the unit Floquet multiplier as $z_1(t_*)$. We also introduce the adjoint Floquet vectors $\{\tilde{z}_i(t_*)\}_{i=1}^n$ such that $\langle \tilde{z}_i(t_*), z_j(t_*)\rangle = \delta_{ij}$ holds, where $\langle \cdot, \cdot \rangle$ denotes an inner product and δ_{ij} the Kronecker's delta symbol. We denote a matrix of the Floquet vectors as $\boldsymbol{Z}(t_*) = (z_2(t_*), z_3(t_*), \cdots, z_n(t_*)) \in \mathbb{R}^{n\times(n-1)}$, and similarly a matrix of the adjoint Floquet vectors as $\tilde{\boldsymbol{Z}}(t_*)$. The Floquet vectors and their adjoints can be obtained by using the numerical method developed for calculating the covariant Lyapunov vectors of continuous dynamical systems [7].

2.3. UCM analysis of periodic motions

Suppose that we obtain sample paths from the system (1) that can be decomposed into N cycles, $\{\boldsymbol{x}_i(t)\}_{i=1}^N$. Each cycle $\boldsymbol{x}_i(t)$ is defined on its own time domain $\mathcal{T}_i \equiv [0, T_i]$. By using the normalization procedure that we explain later, we transform all the cycles so that they have the same length in time. The normalized cycles $\{\tilde{\boldsymbol{x}}_i(\tau)\}_{i=1}^N$ are defined on the same time domain $\tilde{\mathcal{T}} \equiv [0, \tilde{T}]$. We denote the average of the normalized cycles as $\tilde{\boldsymbol{x}}_0(\tau)$.

The performance of a periodic motor task is measured by $\boldsymbol{g}(\boldsymbol{x}) \in \mathbb{R}^m$, which is called a performance variable. When the condition $m < n$ is satisfied, the system (1) has redundant DOFs in generating the identical performance. For example, the position of the center of mass of the body and the end-effector position can be taken as the performance variables [2].

At each moment in the normalized time domain $\tau_* \in \tilde{\mathcal{T}}$, we define the UCM $\mathcal{U}(\tau_*)$ as a subset of Ω whose elements have the same performance variable as that evaluated on the averaged cycle, i.e.

$$\mathcal{U}(\tau_*) \equiv \boldsymbol{g}^{-1}(\boldsymbol{g}(\tilde{\boldsymbol{x}}_0(\tau_*))). \tag{2}$$

The distribution of the data points $\{\tilde{\boldsymbol{x}}_i(\tau_*)\}_{i=1}^N$ gives the degree to which the redundant DOFs are coordinated in stabilizing the performance variable when the state variable is perturbed, and how flexible the coordination is.

The deviation of the performance variable of the ith normalized cycle $\tilde{\boldsymbol{x}}_i(\tau_*)$ from that of the averaged cycle $\tilde{\boldsymbol{x}}_0(\tau_*)$ can be approximated as

$$\boldsymbol{g}(\tilde{\boldsymbol{x}}_i(\tau_*)) - \boldsymbol{g}(\tilde{\boldsymbol{x}}_0(\tau_*)) \approx \boldsymbol{J}(\tau_*)(\tilde{\boldsymbol{x}}_i(\tau_*) - \tilde{\boldsymbol{x}}_0(\tau_*)), \tag{3}$$

where $\boldsymbol{J}(\tau_*) \equiv D\boldsymbol{g}(\tilde{\boldsymbol{x}}_0(\tau_*)) \in \mathbb{R}^{m\times n}$ is the Jacobian matrix of $\boldsymbol{g}$ at $\tilde{\boldsymbol{x}}_0(\tau_*)$. We decompose the deviation $\boldsymbol{\sigma}_i(\tau_*) \equiv \tilde{\boldsymbol{x}}_i(\tau_*) - \tilde{\boldsymbol{x}}_0(\tau_*)$ of the ith normalized cycle from the averaged cycle into two parts as

$$\boldsymbol{\sigma}_i(\tau_*) = \boldsymbol{\sigma}_i^{\mathrm{UCM}}(\tau_*) + \boldsymbol{\sigma}_i^{\mathrm{ORT}}(\tau_*) \tag{4}$$

with

$$\boldsymbol{\sigma}_i^{\mathrm{UCM}}(\tau_*) = (\mathbf{I} - \boldsymbol{J}^+(\tau_*)\boldsymbol{J}(\tau_*))\boldsymbol{\sigma}_i(\tau_*) \tag{5}$$

and

$$\boldsymbol{\sigma}_i^{\mathrm{ORT}}(\tau_*) = \boldsymbol{J}^+(\tau_*)\boldsymbol{J}(\tau_*)\boldsymbol{\sigma}_i(\tau_*), \tag{6}$$

where $\mathbf{I}$ is an identity matrix and $\boldsymbol{J}^+(\tau_*)$ is the Moore-Penrose pseudoinverse of $\boldsymbol{J}(\tau_*)$. The first term $\boldsymbol{\sigma}_i^{\mathrm{UCM}}(\tau_*)$ is the projection of the deviation onto the nullspace of $\boldsymbol{J}(\tau_*)$, which gives a linear approximation to the UCM at $\tilde{\boldsymbol{x}}_0(\tau_*)$. The second term $\boldsymbol{\sigma}_i^{\mathrm{ORT}}(\tau_*)$ is the projection of the deviation onto its orthogonal complement.

We denote the variances of the Euclidean norms of these terms normalized by the number of DOFs in the corresponding subspaces as

$$V^{\mathrm{UCM}}(\tau_*) = \frac{1}{n-m}\frac{1}{N}\sum_{i=1}^{N} \|\boldsymbol{\sigma}_i^{\mathrm{UCM}}(\tau_*)\|^2 \tag{7}$$

and

$$V^{\mathrm{ORT}}(\tau_*) = \frac{1}{m}\frac{1}{N}\sum_{i=1}^{N} \|\boldsymbol{\sigma}_i^{\mathrm{ORT}}(\tau_*)\|^2. \tag{8}$$

The synergy index is then given by

$$S(\tau_*) = \frac{V^{\mathrm{UCM}}(\tau_*) - V^{\mathrm{ORT}}(\tau_*)}{V^{\mathrm{UCM}}(\tau_*) + V^{\mathrm{ORT}}(\tau_*)}. \tag{9}$$

When this index is large, the coordination of the redundant DOFs strongly stabilizes the performance of the task.

2.4. Time-normalization procedure

The time normalization procedure of the stochastic cycles $\{\boldsymbol{x}_i(t)\}_{i=1}^N$ has been performed by rescaling of the waveform $\tilde{\boldsymbol{x}}_i(\tau) = \boldsymbol{x}_i((\tilde{T}/T_i)t)$ in conventional experimental studies. In this study, we propose an alternative procedure that utilizes the asymptotic phase [8], because it gives a more natural equivalence relation on the state space.

Suppose that the noise term of the system (1) is absent for the moment. We can introduce an asymptotic phase $\phi : \mathcal{A} \to [0, T)$ within the basin of attraction $\mathcal{A} \subset \Omega$ of the limit cycle $\boldsymbol{x}_0(t)$, such that

$$\dot{\phi}(\boldsymbol{x}(t)) = 1 \tag{10}$$

holds for any $\boldsymbol{x} \in \mathcal{A}$. The set of states $\mathcal{I}_\phi \equiv \{\boldsymbol{x} \in \mathcal{A} \mid \phi(\boldsymbol{x}) = \phi\}$ that share the same asymptotic phase ϕ, called the isochron [8], produces the same long-term behavior. Namely, initial conditions taken from the same isochron converge to the same trajectory on the limit cycle when the system is unperturbed.

We normalize the stochastic cycles so that the states share the same asymptotic phase, i.e., we define $\tilde{\boldsymbol{x}}_i(\tau) = \boldsymbol{x}_i(t)$ where $\tau = \phi(\boldsymbol{x}_i(t))$ at each moment in the normalized time domain $\tilde{\mathcal{T}} = [0, T]$. We also require that $t_1 < t_2$ if $\tau_1 < \tau_2$, so that the asymptotic phase monotonically increases with time. In the following, we use the phase variable ϕ to express the normalized time instead of τ.

The isochron $\mathcal{I}_{\phi_*}$ is generally difficult to obtain. Therefore, we use a linear approximation to the isochron $\overline{\mathcal{I}}_{\phi_*}$ in evaluating the asymptotic phase of the state. It is known that the adjoint Floquet vector $\tilde{z}_1(\phi_*)$ corresponding to the unit Floquet multiplier is a normal vector of the isochron at $\boldsymbol{x}_0(\phi_*)$ [9]. Therefore, the linear approximation of the isochron is given by

$$\bar{\mathcal{I}}_{\phi_*} \equiv \{\boldsymbol{x} \in \mathcal{A} \mid \langle \tilde{z}_1(\phi_*), \boldsymbol{x} - \boldsymbol{x}_0(\phi_*)\rangle = 0\ \}. \tag{11}$$

2.5. Calculation of the synergy index

We introduce a coordinate transformation $\boldsymbol{x} \mapsto (\phi, \boldsymbol{\rho})$ defined by

$$\boldsymbol{x} = \boldsymbol{x}_0(\phi) + \boldsymbol{Z}(\phi)\boldsymbol{\rho}. \tag{12}$$

If we fix $\phi = \phi_*$ on the right-hand side of Eq. (12), the asymptotic phase of $\boldsymbol{x}$ on the left-hand side is always given by ϕ_* for any $\boldsymbol{\rho}$ within the linear approximation, because, from the biorthogonality of the Floquet vectors and their adjoints, the vectors $\{z_i(\phi_*)\}_{i=2}^{n}$ span a plane $\tilde{I}_{\phi_*}$ that is tangent to the isochron. Hence, the conditional covariance of $\boldsymbol{\rho}$ given $\phi = \phi_*$, which we denote as $\boldsymbol{\Sigma}_{\phi_*} \in \mathbb{R}^{(n-1)\times(n-1)}$, completely determines $V^{\mathrm{UCM}}(\phi_*)$ and $V^{\mathrm{ORT}}(\phi_*)$ in the linear approximation.

As we will discuss in detail in our future publication, we can derive approximate expressions for the conditional mean $\boldsymbol{\mu}_{\phi_*} \in \mathbb{R}^{n-1}$ and the covariance $\boldsymbol{\Sigma}_{\phi_*}$ up to the first order in the noise intensity ϵ as follows. Firstly, the conditional mean is approximated as

$$\boldsymbol{\mu}_{\phi_*} \approx \boldsymbol{0}. \tag{13}$$

Therefore, the average cycle $\tilde{\boldsymbol{x}}_0(\phi)$ of the normalized cycles is nearly identical with the unperturbed limit-cycle orbit $\boldsymbol{x}_0(\phi)$. Secondly, the conditional covariance approximately satisfies

$$\boldsymbol{\Sigma}_{\phi_*} \approx \boldsymbol{A}\boldsymbol{\Sigma}_{\phi_*}\boldsymbol{A}^{\mathrm{T}} + \epsilon^2\boldsymbol{\Pi}_{\phi_*}, \tag{14}$$

where

$$\boldsymbol{A} = \exp\int_0^T \boldsymbol{\Lambda}(\phi_* + \phi)d\phi = \exp\int_0^T \boldsymbol{\Lambda}(\phi)d\phi, \tag{15}$$

$$\boldsymbol{\Pi}_{\phi_*} = \boldsymbol{A}\left(\int_0^T \hat{\boldsymbol{\Lambda}}_{\phi_*}(\phi)\tilde{\boldsymbol{Z}}^{\mathrm{T}}(\phi_*+\phi)\boldsymbol{P}(\boldsymbol{x}_0(\phi_*+\phi))\right.$$
$$\left.\boldsymbol{P}^{\mathrm{T}}(\boldsymbol{x}_0(\phi_*+\phi))\tilde{\boldsymbol{Z}}(\phi_*+\phi)\hat{\boldsymbol{\Lambda}}^{\mathrm{T}}_{\phi_*}(\phi)d\phi\right)\boldsymbol{A}^{\mathrm{T}}, \tag{16}$$

and

$$\boldsymbol{\Lambda}(\phi) = \tilde{\boldsymbol{Z}}(\phi)^{\mathrm{T}}D\boldsymbol{f}(\boldsymbol{x}_0(\phi))\boldsymbol{Z}(\phi), \tag{17}$$

$$\hat{\boldsymbol{\Lambda}}_{\phi_*}(\phi) = \exp\int_0^{\phi} -\boldsymbol{\Lambda}(\phi_* + u)du. \tag{18}$$

Here, the matrix $\boldsymbol{A}^{\mathrm{T}}$ is the transpose of $\boldsymbol{A}$. By permutating the column vector of $\boldsymbol{Z}(\phi)$, we can assume, without loss of generality, that the matrix $\boldsymbol{A}$ has a block diagonal form

$$\boldsymbol{A} = \mathrm{diag}(\bar{\lambda}_2, \cdots, \bar{\lambda}_r, \bar{\boldsymbol{\Lambda}}_{r+1}, \cdots, \bar{\boldsymbol{\Lambda}}_{r+c}), \tag{19}$$

whose diagonal elements correspond to the Floquet multipliers. Here, $\bar{\lambda}_i \in \mathbb{R}$, $(i = 2, \cdots, r)$, $\bar{\boldsymbol{\Lambda}}_i \in \mathbb{R}^{2\times 2}$, $(i = r+1, \cdots, r+c)$, and $r + 2c = n$ holds.

Once $\boldsymbol{\Sigma}_{\phi_*}$ is obtained, we can calculate the synergy index using the following relations:

$$V^{\mathrm{UCM}}(\phi_*) = \frac{\mathrm{tr}(\boldsymbol{Z}^{\mathrm{T}}(\phi_*)(\mathbf{I} - \boldsymbol{J}^+(\phi_*)\boldsymbol{J}(\phi_*))\boldsymbol{Z}(\phi_*)\boldsymbol{\Sigma}_{\phi_*})}{n-m}, \tag{20}$$

$$V^{\mathrm{ORT}}(\phi_*) = \frac{1}{m}\mathrm{tr}(\boldsymbol{Z}^{\mathrm{T}}(\phi_*)\boldsymbol{J}^+(\phi_*)\boldsymbol{J}(\phi_*)\boldsymbol{Z}(\phi_*)\boldsymbol{\Sigma}_{\phi_*}), \tag{21}$$

where $\mathrm{tr}(\cdot) \in \mathbb{R}$ is the trace of a matrix.

3. Example

As a simple example, we consider an analytically tractable dynamical system for a two-dimensional state variable $\boldsymbol{x} = (x, y)^{\mathrm{T}}$ given by

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} x - (2\pi+1)y - (x^2+y^2)(x-y) \\ (2\pi+1)x + y - (x^2+y^2)(x+y) \end{pmatrix}. \tag{22}$$

This oscillator, called the Stuart-Landau oscillator [9], is a generic example of stable periodic motions because it arises universally in the vicinity of the supercritical Hopf bifurcation. It has a stable limit-cycle orbit as shown in Fig. 1. We set the zero point of the asymptotic phase as $\phi((1,0)^{\mathrm{T}}) = 0$.

When we consider the case $\boldsymbol{P}(\boldsymbol{x}) = \mathbf{I}$ and $\boldsymbol{g}(\boldsymbol{x}) = x$, where the performance variable is a scalar function, the synergy index can be calculated as

$$S(\phi) = \sin 2\phi, \tag{23}$$

by the proposed theory. This result is compared with that of direct numerical simulations of the stochastic differential equation (1). As shown in Fig. 2, good agreement between the results is observed.

4. Summary

We have proposed a theory that relates the synergy index of a periodic motion to the Floquet eigenvalues and eigenvectors of the limit-cycle orbit, and applied it to a simple mathematical model of noisy limit-cycle dynamics. The theory would be useful in performing the UCM analysis for dynamical systems models of body movement.

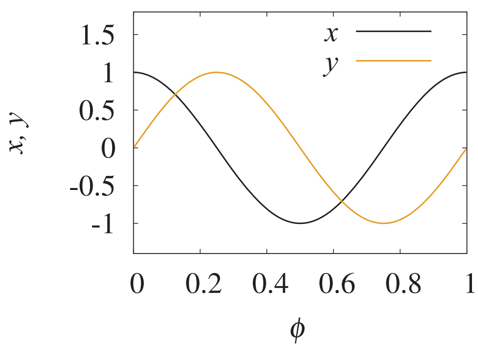

Figure 1: Limit-cycle solution of a Stuart-Landau oscillator.

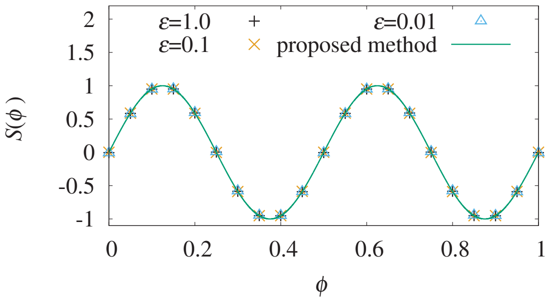

Figure 2: Comparison of the synergy index obtained by the proposed theory and by direct numerical simulations performed for several noise intensities.

Acknowledgments

Financial support by MEXT/JSPS KAKENHI (No. 25540108, 26103510, 26120513, 15J12045) is gratefully acknowledged. m

References

[1] N. A. Bernstein, ”The coordination and regulation of movements”, *Pergamon Press*, 1967.

[2] M. L. Latash, J. P. Scholz and G. Schöner, ”Toward a new theory of motor synergies”, *Motor Control*, vol.11, pp.276–308, 2007.

[3] J. P. Scholz and G. Schöner, ”The uncontrolled manifold concept: identifying control variables for a functional task”, *Exp. Brain Res.*, vol.126, pp.289–306, 1999.

[4] M. Santello and C. E. Lang, ”Are movement disorders and sensorimotor injuries pathologic synergies? When normal multi-joint movement synergies become pathologic”, *Front. Hum. Neurosci.*, vol.8, pp.1–13, 2015.

[5] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner and G. Vattay, ”Chaos: Classical and Quantum”, *Niels Bohr Institute*, 2012.

[6] C. Gardiner, ”Stochastic Methods: A Handbook for the Natural and Social Sciences”, *Springer*, 2009.

[7] F. Ginelli, H. Chaté, R. Livi and A. Politi, ”Covariant Lyapunov vectors”, *J. Phys. A: Math. Theor.*, vol.46, 2013.

[8] G. B. Ermentrout and D. H. Terman, ”Mathematical Foundatations of Neuroscience”, *Springer*, 2010.

[9] Y. Kuramoto, ”Chemical Oscillations, Waves, and Turbulence”, *Dover Publications*, 2003.