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Stability and sensitivity of synchronized states in a network of symmetrically coupled nonlinear oscillators for generating gait patterns

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Abstract—A coupled-oscillator model for the central pattern generator proposed by Golubitsky *et al.* [1], which can exhibit various synchronized states that correspond to typical quadruped gaits, is studied. The stability and sensitivity of the synchronized states are quantified by the Lyapunov exponents and the associated Lyapunov vectors. It is shown that the stability of the synchronized state depends on the gaits, and the Lyapunov vectors reflect the symmetry of the gaits. The asymptotic phase response of the model to external perturbations is characterized by the adjoint Lyapunov vector associated with the zero Lyapunov exponent. Phase response properties of the gait measured by direct numerical stimulations reasonably agree with the adjoint Lyapunov vectors.

1. Introduction

It is considered that the gaits (walking patterns) of animals and insects are generated by the central pattern generators (CPGs) in their neural systems [2]. Although physiological details of the CPG are still under investigation, various mathematical models for the CPG have been developed [1]. In particular, networks of mutually interacting nonlinear oscillators, which can exhibit a variety of stable rhythmic patterns, have been studied as prototypical models for CPGs. Such mathematical models are also applied in controlling the locomotion of artificial robots with multiple legs [3].

In this study, we analyze a simple coupled-oscillator model for the CPG introduced by Golubitsky *et al.* [1], which consists of symmetrically coupled identical oscillators and is capable of reproducing several representative gaits observed in real animals. Reflecting its symmetric coupling networks, this model exhibits symmetric synchronized states. Namely, the oscillators settle in steady synchronized states with particular phase relations with the other oscillators, which can be interpreted as rhythmic gaits.

We characterize the stability and sensitivity of the synchronized states by the Lyapunov exponents and the associated Lyapunov vectors. Using recently developed numerical methods [4, 5], we calculate the Lyapunov exponents, covariant or characteristic Lyapunov vectors (simply called Lyapunov vectors in the following), and the adjoint Lyapunov vectors for several representative gaits and analyze their properties.

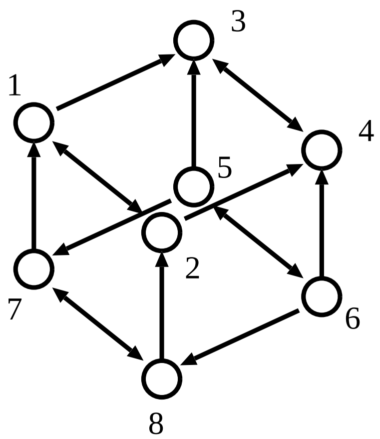

Figure 1: Coupled-oscillator model for generating quadruped gaits proposed by Golubitsky *et al.* [1]. Variables x_1–x_4 of the oscillators 1–4 correspond to the 4 legs.

2. Model

We consider a coupled-oscillator model for the CPG introduced by Golubitsky *et al.* [1], which consists of $4n$ identical oscillators for $2n$-legged animals. Each oscillator is described by the FitzHugh-Nagumo neuron model,

$$\dot{x}(t) = c\left(x + y - \frac{1}{3}x^3\right) \equiv f_1(x, y),$$
$$\dot{y}(t) = -\frac{1}{c}(x - a + by) \equiv f_2(x, y), \qquad (1)$$

where the variable x corresponds to the membrane potential of the neuron and y represents its activation level. Denoting the i-th oscillator state as (x_i, y_i) where $i = 1, 2, ..., 4n$, the CPG model is given by the following coupled dynamical equations:

$$\dot{x}_i(t) = f_1(x, y) + \alpha(x_{i-2} - x_i) + \gamma(x_{i+\epsilon} - x_i),$$
$$\dot{y}_i(t) = f_2(x, y) + \beta(y_{i-2} - y_i) + \delta(y_{i+\epsilon} - y_i), \qquad (2)$$

where

$$\epsilon = 1 \text{ (when } i \text{ is odd)}, \ -1 \text{ (when } i \text{ is even)}, \tag{3}$$

and the oscillator index is considered in modulo $4n$, i.e., $i-2 = 4n+i-2$ for $i = 1, 2$.

We consider the case with $n = 2$, namely, the gaits of quadrupeds with 4 legs. See Fig. 1 for a schematic of the model. The variables x_1, x_2, x_3, and x_4 of the oscillators 1–4 are interpreted as the movement of the left-rear foot, right-rear foot, left-front foot, and right-front foot, respectively. The parameters a, b, and c specify the dynamics of individual oscillator and are fixed at $a = 0.02$, $b = 0.2$, and $c = 0.44$. The parameters α, β, γ, and δ specify mutual coupling between the oscillators. As shown in [1], by setting these coupling parameters appropriately, this model can exhibit synchronized states corresponding to the following gaits: pace, bound, trot, jump, and walk. See [1] for the details.

In this article, we focus only on the trot and walk gaits. In the trot gait, the 8 oscillators form 2 clusters with $x_1 = x_4 = x_5 = x_8$ and $x_2 = x_3 = x_6 = x_7$ (and similarly for y). Thus, the left-rear foot and the right-front foot form a pair, and the left-front foot and the right-rear foot form another pair. Similarly, in the walk gait, the 8 oscillators form 4 clusters with $x_1 = x_6$, $x_2 = x_5$, $x_3 = x_8$, and $x_4 = x_7$ (and similarly for y). See Table 1 for the parameter values corresponding to various gaits, and Fig. 2 for the dynamics of the oscillators 1–4 in the trot and walk states.

Gait	α	β	γ	δ
Pace	0.025	0.02	-0.01	-0.012
Trot	-0.02	-0.002	-0.025	0.015
Bound	-0.01	-0.0102	0.025	0.02
Jump	-0.02	0.01	0.025	0.015
Walk	-0.01	0.0102	-0.025	0.02

Table 1: Parameter values for the trot and walk gaits. Taken from Golubitsky *et al.* [1].

3. Lyapunov exponents and vectors

The Lyapunov exponents quantify linear growth rates of small variations from a given trajectory [4, 5, 6]. In m-dimensional dynamical systems, there exist m independent directions and therefore m Lyapunov exponents. The Lyapunov vectors give the directions associated with the Lyapunov exponents. When the system exhibits periodic dynamics, these quantities are essentially equivalent to the Floquet exponents and the associated Floquet vectors of the periodic orbit [6].

We consider a m-dimensional continuous dynamical system described by

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{G}(\boldsymbol{X}). \tag{4}$$

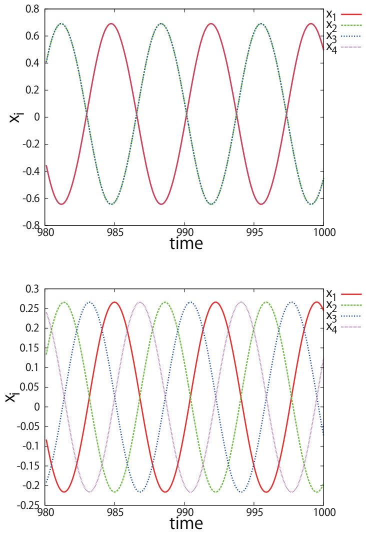

Figure 2: Examples of the oscillator dynamics. Trot (top) and walk (bottom).

For the present CPG model, the state vector is given by

$$\boldsymbol{X} = (x_1, y_1, \cdots, x_8, y_8) \tag{5}$$

and the dimension of the model is $m = 16$. The linearized dynamics of the infinitesimal variation $\boldsymbol{u}(t)$ from the trajectory $\boldsymbol{X}(t)$ is given by

$$\dot{\boldsymbol{u}}(t) = \mathrm{J}(\boldsymbol{X})\boldsymbol{u} \tag{6}$$

where J is a Jacobian matrix of $\boldsymbol{G}(\boldsymbol{X})$. The fundamental matrix $\mathrm{M}(t)$, whose columns are given by linearly independent solutions of Eq. (6), obeys

$$\dot{\mathrm{M}}(t) = \mathrm{J}(\boldsymbol{X})\mathrm{M}, \tag{7}$$

and the solution to Eq. (6) can be expressed as

$$\boldsymbol{u}(t_2) = \mathrm{F}(t_1, t_2)\boldsymbol{u}(t_1), \tag{8}$$

where the propagator is given by [5]

$$\mathrm{F}(t_1, t_2) = \mathrm{M}(t_2)\mathrm{M}(t_1)^{-1}. \tag{9}$$

The Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and the associated Lyapunov vectors $\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \cdots, \boldsymbol{v}_n(t)$ satisfy

$$\lim_{t \to \pm\infty} \frac{1}{t} \ln \|\mathrm{F}(t_1, t_1 + t)\boldsymbol{u}(t_1)\| = \pm\lambda_j \tag{10}$$

if $\boldsymbol{u}(t_1) \parallel \boldsymbol{v}_j(t_1)$ for any t_1 ($j = 1, 2, ..., m$). Thus, if a small variation $\boldsymbol{u}(t_1)$ that is parallel to the Lyapunov vector $\boldsymbol{v}_j(t_1)$ is given to the state point $\boldsymbol{X}(t_1)$ at $t = t_1$, it grows exponentially with a rate λ_j as

$$\|\mathrm{F}(t_1, t_1 + t)\boldsymbol{u}(t_1)\| \sim e^{\lambda_j t}. \tag{11}$$

Following Kuptsov and Parlitz [5], an adjoint propagator

$$G(t_1, t_2) = F(t_1, t_2)^{-T} \tag{12}$$

can also be introduced, where $-T$ denotes matrix inversion and transpose. The adjoint Lyapunov vectors $\boldsymbol{w}_1(t), \boldsymbol{w}_2(t), \cdots, \boldsymbol{w}_n(t)$ then satisfy

$$\lim_{t\to\pm\infty} \frac{1}{t} \ln \|G(t_1, t_1 + t)\boldsymbol{u}(t_1)\| = \mp\lambda_j \tag{13}$$

if $\boldsymbol{u}(t_1) \parallel \boldsymbol{w}_j(t_1)$ for any t_1 $(j = 1, 2, ..., m)$. The Lyapunov and adjoint vectors $\{\boldsymbol{v}_1(t), \cdots, \boldsymbol{v}_m(t)\}$ and $\{\boldsymbol{w}_1(t), \cdots, \boldsymbol{w}_m(t)\}$ form a biorthogonal basis and satisfy

$$\boldsymbol{v}_i(t) \cdot \boldsymbol{w}_j(t) = \delta_{ij} \tag{14}$$

for $i, j = 1, 2, ..., m$. These adjoint Lyapunov vectors can be used to project the perturbations onto the direction of the corresponding Lyapunov vectors, and therefore they characterize the "sensitivity" of the limit-cycle orbit to given perturbations.

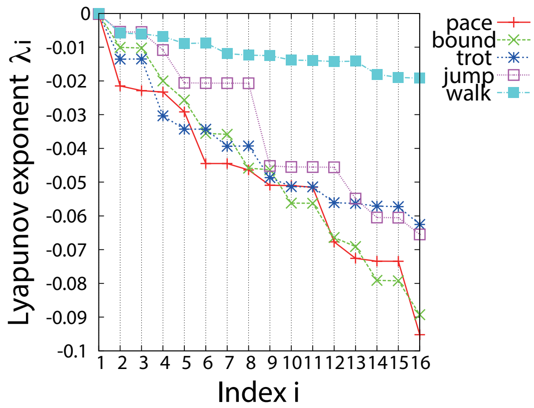

Figure 3: Lyapunov exponents for each gait in Table 1.

4. Results

We use the numerical algorithm proposed by Kuptsov and Parlitz [5] to calculate the Lyapunov exponents and vectors. Figure 3 shows the Lyapunov exponents for the gaits shown in Table 1. The first Lyapunov exponent λ_1 vanishes for all gaits and all other exponents are negative, i.e., all gaits are linearly stable. The values of the second and other Lyapunov exponents differ from gait to gait, which shows that the stability of the synchronized state depends on the gait. It can also be seen that the Lyapunov exponents decrease with i stepwisely, indicating that some of the exponents are degenerate. This reflects the symmetry in the synchronized dynamics of the oscillators.

We first focus on the second and other Lyapunov vectors $\boldsymbol{v}_2, \boldsymbol{v}_3, \cdots, \boldsymbol{v}_n$. These vectors correspond to the "amplitude deviations" away from the synchronized state and perturbations given in these directions decay exponentially. The Lyapunov vectors associated with the negative Lyapunov exponents with the smallest magnitude characterize the perturbations that decay most slowly, i.e., perturbations that affect the stability of the gait most strongly. Figure 4 shows the Lyapunov vectors $\boldsymbol{v}_2(t)$ and $\boldsymbol{v}_3(t)$ for the trot gait (a, b) and walk gait (c, d). As can be seen from Fig. 1, the 2nd and 3rd Lyapunov exponents are degenerate, i.e., $\lambda_2 = \lambda_3$, except for the pace gait. Therefore, the corresponding Lyapunov vectors $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are not uniquely determined, and external perturbations in the 2-dimensional space spanned by $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ decay with the same rate.

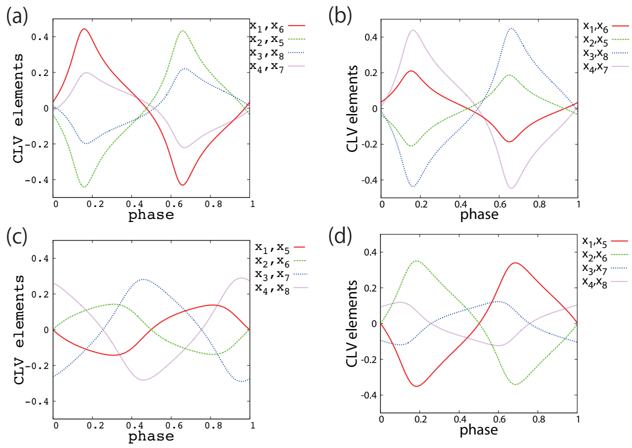

Figure 4: Lyapunov vectors $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ associated with the Lyapunov exponents $\lambda_2 = \lambda_3$, plotted as functions of the oscillation phase. (a) $\boldsymbol{v}_2$ for trot, (b) $\boldsymbol{v}_3$ for trot, (c) $\boldsymbol{v}_2$ for walk, and (d) $\boldsymbol{v}_3$ for walk.

We now focus on the Lyapunov vector $\boldsymbol{v}_1$ associated with $\lambda_1 = 0$, which is tangent to the limit-cycle orbit. External perturbations given in this direction do not grow or decay exponentially. Therefore, this direction is neutrally stable and corresponds to the "phase" direction of the synchronized state, i.e., the collective limit-cycle oscillation of the whole system. Figure 5 shows the Lyapunov vectors $\boldsymbol{v}_1(t)$ and the adjoint Lyapunov vectors $\boldsymbol{w}_1(t)$ for the trot gait. Similarly, Fig. 6 shows the Lyapunov vectors $\boldsymbol{v}_1(t)$ and the adjoint Lyapunov vectors $\boldsymbol{w}_1(t)$ for the walk gait.

The effect of external perturbations that are parallel to $\boldsymbol{v}_1(t)$ remains as a constant phase shift in the collective oscillation of the system. More explicitly, if the system is kicked by an instantaneous perturbation $\boldsymbol{p}$ at time t, the phase shift is proportional to the projection of $\boldsymbol{p}$ onto the direction of $\boldsymbol{v}_1(t)$, i.e., the product of the adjoint Lyapunov vector $\boldsymbol{w}_1(t)$ and $\boldsymbol{p}$ [7]. Figure 7 compares the adjoint Lyapunov vector $\boldsymbol{w}_1(t)$ and the phase response of the gait measured by directly perturbing each of the x_1-x_4 variables with an impulse of intensity 0.02 for the walk gait. The results agree reasonably when appropriately rescaled, indicating that the adjoint Lyapunov vector correctly characterizes the phase response properties of the synchronized state.

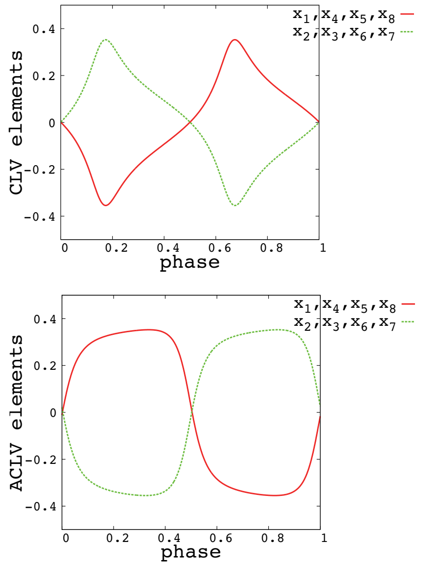

Figure 5: Lyapunov vectors $\boldsymbol{v}_1$ (top) and the adjoint Lyapunov vectors $\boldsymbol{w}_1$ (bottom) associated with $\lambda_1 = 0$ for the trot gait. Only x components are shown for one period of oscillation.

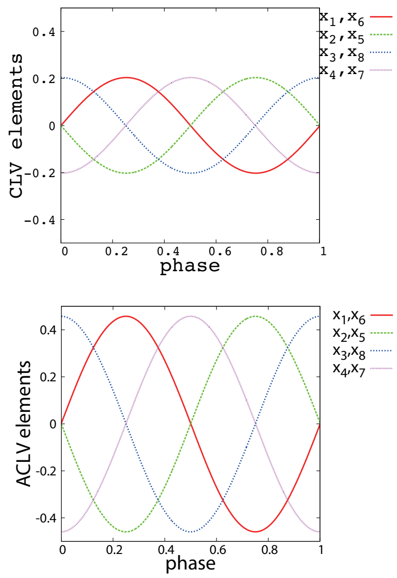

Figure 6: Lyapunov vectors $\boldsymbol{v}_1$ (top) and the adjoint Lyapunov vectors $\boldsymbol{w}_1$ (bottom) associated with $\lambda_1 = 0$ for the walk gait. Only x components are shown for one period of oscillation.

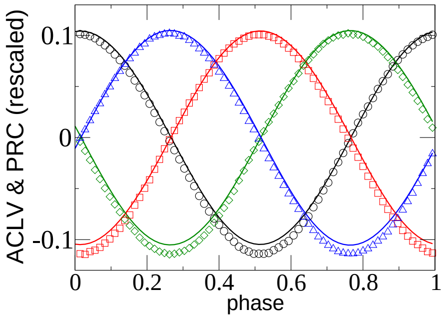

Figure 7: Comparison of the adjoint Lyapunov vector $\boldsymbol{w}_1$ and the phase response function obtained by direct numerical simulations. The axes are appropriately rescaled. Only $x_{1,2,3,4}$ components are shown.

5. Summary

We have characterized the stability and sensitivity of the synchronized states in a coupled-oscillator model for gait generation using the Lyapunov and adjoint Lyapunov vectors. More details of the Lyapunov vectors, in particular symmetry properties, will be discussed at the conference.

Acknowledgments

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