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An evolving network based on a threshold graph and estimation of its evolution process

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Abstract— In this paper, we propose a simple evolving network model based on a threshold graph. In the proposed network model, a node has its own state, and connections to other nodes are determined by the state. Because the states of nodes evolve with time in our network model, the connections between nodes also change with time. We also propose a method for reconstructing the evolution processes of the states of nodes in the networks only from the information of the structures of networks at each time. Numerical experiments clearly show that the proposed method can reconstruct the evolution dynamics.

1. Introduction

In the real world, various systems are described as a network, for example human relationships, synaptic connections in neural systems, the world wide web [2], ecological food webs [1], and so on. These real networks generally consist of a number of nodes which are intricately connected with each other. Even though the real networks have such complex structures, recent researches on these real complex networks have clarified that underlying common structures exist, such as a small-world structure and a scale-free structure. To analyze real complex networks, several network models that realize these structural features have been proposed [4, 5].

On the other hand, in recent years, temporal evolution of networks is focused [6, 7]. In Ref. [6], temporal changes of e-mail networks were discussed: sent and received e-mails were observed for about four months and e-mail networks were analyzed. It was then reported that the e-mail networks keep a power-law degree distribution each day, but hub nodes dynamically change from day to day. As in the case of the e-mail network, many real networks might evolve over time, changing their structures. However, the conventional network models cannot explain such dynamical features of the real complex networks because those models mainly focus on the static features of real networks at a given time.

In this paper, we first propose a simple evolving network model in which structural features of networks, such as degree distribution, are kept through time but connections of nodes dynamically change over time. Our network model is based on a threshold graph in which nodes have their own weights and connections between the nodes are determined by the weights [8, 9, 10]. Although the threshold graph is a simple network model, it can realize various types of complex networks. In our proposed network model, nodes have their own states instead of the weight in the threshold graph, and the states are evolved by intrinsic dynamics of the nodes over time. The connections between nodes are decided by the following rule: if the states between two nodes are "close", they are connected, otherwise disconnected. Because the states evolve with time and existence of connections depends on "closeness" between the states of the nodes, generated networks always change dynamically through time. Our model assumes the evolution of networks might obey underlying, usually unknown, dynamics. We show that our assumption is valid from the viewpoint of the stationarity of structural features of networks through time and dynamical changes of degree of nodes.

In the second part of the paper, we propose a method for reconstructing temporal changes of the states of nodes only from adjacency relations between the nodes at each time. In numerical simulations, we show that our method is effective for uncovering the hidden rule and reconstructing the evolution processes from the network structures.

2. Evolving network model

In our model, the ith node $v(i)$ $(i = 1, \ldots, n)$ in a network has its own state $\boldsymbol{x}_i(t) \in R^m$, where $\boldsymbol{x}_i(t)$ is an m-dimensional state vector at time t. Let us assume that the state of the ith node $\boldsymbol{x}_i(t)$ is evolved by the following dynamics:

$$\boldsymbol{x}_i(t+1) = \boldsymbol{F}_i(\boldsymbol{x}_i(t)), \quad (1)$$

where $\boldsymbol{F}_i$ is an m-dimensional map for the ith node. For the sake of simplicity, in the following, we only consider 1-dimensional discrete time dynamical systems, namely $m = 1$. We assume also that all nodes obey the same dynamical rule. However, the method described below can be applied to higher dimensional systems and continuous time dynamical systems.

At time t $(t = 1, \ldots, T)$, two nodes $v(i)$ and $v(j)$ are connected if the Euclidean norm $d_{ij}(t) = |x_i(t) - x_j(t)|$

is smaller than a threshold θ, but if not, $v(i)$ and $v(j)$ are disconnected. Because whether two nodes are connected or not depends on their states, the connections dynamically change as the states evolve. Figure 1 is a schematic diagram of the proposed network model. At time t, the nodes $v(i)$ and $v(k)$ are connected because $d_{ik}(t) < \theta$, while $v(i)$ and $v(j)$ are not connected because $d_{ij}(t) > \theta$ (the left side of Fig. 1). However, at time $t+1$, the states of nodes are updated by F and thereby the connections change (the right side of Fig. 1). Thus, a network changes its structure with time because the states of the nodes change with time.

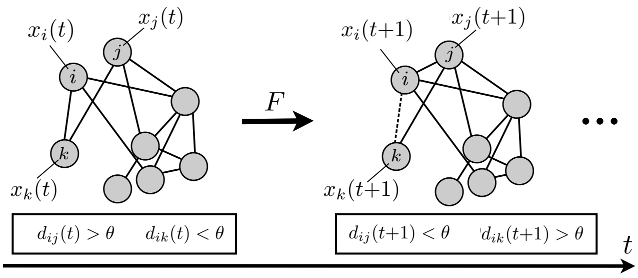

Figure 1: The proposed network model. The network structures change with time t becasuse states of the nodes change with time.

From Ref. [8], the degree distribution $p(k)$ of the threshold graph is described by

$$p(k) = \rho(w)\frac{dw}{dk}, \tag{2}$$

where k is the degree, w is the weights of nodes, and $\rho(w)$ is the distribution of the weight. Equation (2) indicates that the degree distribution of networks depends only on the distribution of weights $\rho(w)$. In our model, if the dynamical rules of all nodes are stationary, the degree distribution is invariant even if the states of the nodes are evolved with time. Thus, in the case, the degree distribution is kept even if the states are updated over time. However, the degree of each node changes with time.

3. Method for reconstructing evolution process

In the following sections, under the assumption that some dynamical rules exist in the evolution of networks, we propose a method for reconstructing $\boldsymbol{x}_i(t)$, time series of the state of the ith node $v(i)$, only from network structures. We assume the following situations: we only have networks, namely adjacency matrices, for any time t and then try to estimate the unknown state of $v(i)$, $\boldsymbol{x}_i(t)$, only from the adjacency matrices.

From a network at time t $(1 \leq t \leq T)$, we reconstruct the time series of the states of the ith node $\boldsymbol{x}_i(t)$. To reconstruct $\boldsymbol{x}_i(t)$, we applied the method proposed by Hirata et al. [11] which reproduces an original time series only from its recurrence plot [13, 14]. The recurrence plot is an image generated from an attractor of a nonlinear dynamical system. From Ref. [11], once the recurrence plot is given, the unknown distances d_{ij} between any two points on the attractor can be estimated. Then, by the classical multidimensional scaling (CMDS) [12], coordinate values of the points are reconstructed so that the estimated distance relations are satisfied. In our method, because the adjacency matrix of a network corresponds to the recurrence plot, we can directly apply the method in Ref. [11] to the adjacency matrix and can obtain the states of nodes only from the adjacency matrix.

To reconstruct a state time series $\boldsymbol{x}_i(t)$, firstly distances between any two nodes are estimated. Let $D(t) = \{d_{ij}(t)\}$ be the distance matrix between two nodes $v(i)$ and $v(j)$ in a network at time t. If two nodes $v(i)$ and $v(j)$ are adjacent each other, the distance $d_{ij}(t)$ is estimated by the following equation [11]:

$$d_{ij}(t) = 1 - \frac{|G_i(t) \cap G_i(t)|}{|G_i(t) \cup G_i(t)|}, \tag{3}$$

where $G_i(t)$ is a set of indices of adjacent nodes of the node $v(i)$ at time t, $|G|$ is the number of components in the set G, and $\cap$ and $\cup$ are the union and the intersection of two sets. If two nodes are not adjacent, the distance between them is evaluated by the shortest path length calculated from the distances between the adjacent nodes. We used the Dijkstra algorithm to calculate the shortest path length.

Next, the CMDS is applied to the distance matrix $D(t)$. The procedure of the CMDS is mainly divided into two parts [12]: (i) centering a matrix $D^{(2)}(t)$ by

$$A(t) = -\frac{1}{2}JD^{(2)}(t)J, \tag{4}$$

where J is the centering matrix whose diagonal elements are $1 - 1/n$ and other elements are $1/n$, n is the number of nodes in a network, and $D^{(2)}(t) = \{d_{ij}^2\}$. (ii) Applying its eigenvector decomposition as follows.

$$A(t) = U(t)\Lambda(t)U^{\mathsf{T}}(t), \tag{5}$$

where $\Lambda(t) = \mathrm{diag}(\lambda_1(t), \ldots, \lambda_m(t))$, $U(t) = (\boldsymbol{u}_1(t) \ \cdots \ \boldsymbol{u}_m(t))$, $\boldsymbol{u}_i(t) = (u_{i1}(t) \ u_{i2}(t) \ \cdots \ u_{in}(t))^{\mathsf{T}}$ and $\lambda_i(t)$ and $\boldsymbol{u}_i(t)$ are the ith eigenvalue and the ith eigenvector of the matrix $A(t)$. Here, let the coordinate value, or the state, of the node $v(i)$ be $\boldsymbol{x}_i(t)$ and $X(t) = (\boldsymbol{x}_1(t) \ \cdots \ \boldsymbol{x}_n(t))^{\mathsf{T}}$. From the relationship between the innerproduct and the distance [12], $X(t)$ is described by

$$X(t) = U(t)\Lambda(t)^{1/2}, \tag{6}$$

where $\Lambda(t)^{1/2} = \mathrm{diag}(\sqrt{\lambda_1(t)}, \ldots, \sqrt{\lambda_m(t)})$. Then, the rank of $X(t)$ corresponds to the dimension m of the states of nodes. Thus, the estimated state of the node $v(i)$ at time t is written by

$$\hat{\boldsymbol{x}}_i(t) = (\sqrt{\lambda_1(t)}u_{1i}(t) \ \sqrt{\lambda_2(t)}u_{2i}(t) \ \cdots \ \sqrt{\lambda_3(t)}u_{mi}(t))^{\mathsf{T}}. \tag{7}$$

By applying the above mentioned method to the networks for all time t, we can reconstruct the time series of states of nodes $\hat{\boldsymbol{x}}_i(t)$.

4. Simulation settings

To evaluate our method, we conducted numerical experiments in the following manner. As evolution rules in our model, we used three maps: the logistic map [15], $x_i(t+1) = 4x_i(t)(1 - x_i(t))$, the auto regressive (AR) model [16], $x_i(t+1) = 0.9x_i(t) + \epsilon(t)$, and the random series, $x_i(t) = \epsilon(t)$. In the AR model and the random series, $\epsilon(t)$ is a random number which obeys the Gaussian distribution whose average μ and variance σ^2.

At first, we generated networks from our proposed network model. As initial states of nodes, we assigned a uniform random number in $[0, 1]$ to the states of nodes $v(i)$, $x_i(0)$. We then evolved the state $x_i(t)$ by F until time T. As time evolves, the states change and the structures of networks also change. After that we obtained T networks. In our experiment, we determined the threshold θ so that the number of connections in the initial network becomes $p\%$ of the total links in the complete graph with n nodes. We set the number of nodes $n = 100$ and evolved the network until $T = 100$.

5. Results

In our method, we first estimated distances among all the nodes by Eq.(3). Next, we applied the CMDS to the distance matrix $D(t)$. Then, the nodes are arranged into an m-dimensional Euclidean space so that the distances among them are satisfied. The dimension m of the states is determined by the number of nonzero eigenvalues $\lambda_s(t)$, $(s = 1, \ldots, m$ and $\lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq \lambda_m(t) > 0)$.

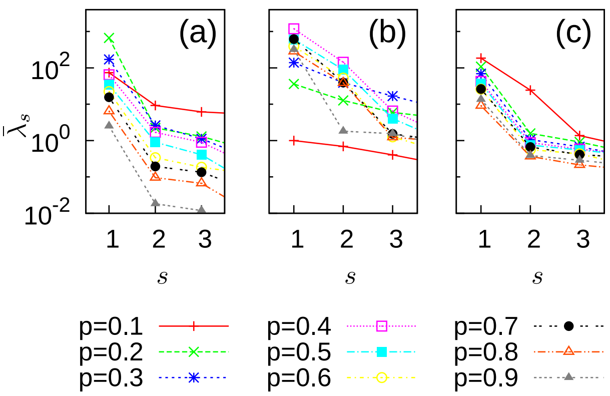

Figure 2: The temporal average of eigenvalues $\bar{\lambda}_s$ for s. The eigenvalues are calculated from the networks generated from our model with (a) the logistic map, (b) the AR model, and (c) the random series. Vertical axes are plotted in a log-scale.

Figure 2 shows temporal average of the eigenvalues $\bar{\lambda}_s = \frac{1}{T}\sum_{t=1}^{T}\lambda_s(t)$. In Fig. 2, the parameter p was changed. From Fig.2, the first eigenvalue $\bar{\lambda}_1$ is sufficiently larger than the other eigenvalues for all p. It indicates that it is enough to describe the states of nodes in a 1-dimensional Euclidean space. Thus the estimated state time series of the node $v(i)$ is written by the following equation from Eq.(7):

$$\hat{x}_i(t) = \sqrt{\lambda_1(t)}u_{1i}(t) \tag{8}$$

Next, we investigated how precisely we can reconstruct the original time series. Figure 3 shows the results for the logistic map, the AR model, and the random series. In Fig. 3, the correlation coefficient between the original time series and the reconstructed time series is calculated for several values of p (see also Appendix for details of calculation of the correlation coefficients). The correlation coefficient is averaged over all nodes for p. From Fig. 3, when the threshold value p is sufficiently large, we can reconstruct original time series only from network structures for all types of networks $(20\% < p < 90\%)$. The results indicate that when some dynamical rules exist in real networks, we can reconstruct them by our strategy and thereby we can analyze the evolution of networks toward its prediction, its control, and its modeling.

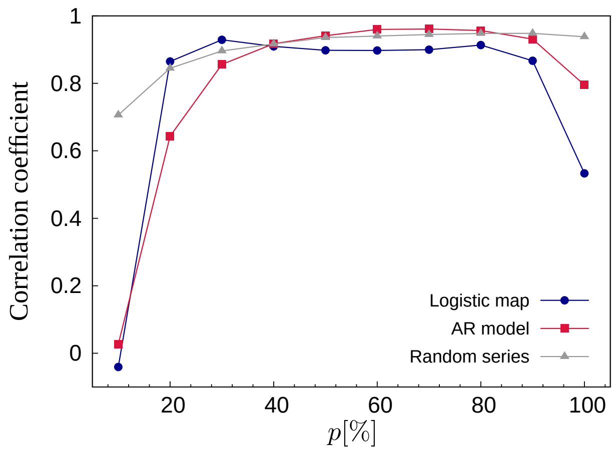

Figure 3: Correlation coefficients between the reconstructed time series and corresponding original time series are calculated for several values of p. The logistic map (dark blue), the AR model (red), and the random series (gray) are used to produce the original network. In the AR model and the random series, the averages and the variances of $\epsilon(t)$ is set to zero and unity.

6. Conclusion

In this paper, we proposed a simple evolving network model based on a threshold graph [8]–[10]. In our model, the node has its own state and the states evolve with time. Because the connections between the nodes are determined by the Euclidean norm between the states of the nodes, the network structures also change with time. We also proposed a method for reconstructing the time series of the nodes only from the networks at each time. In numerical simulations, we applied our reconstruction method to some networks in which the states of nodes are evolved by a dynamical rule. As a result, we could reconstruct the time series of the nodes using our reconstruction method. The results indicate that if some dynamical rules exist in real networks, we can analyze how real complex networks were in the past and how they will be in the future from the viewpoint of the dynamical system theory.

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Appendix: Calculating the correlation coefficient

In our method, we used CMDS to arrange the nodes in an m-dimensional space. In the CMDS, nodes are arranged to satisfy distance relations between them, but the signatures of coordinate vectors are arbitrary determined. Thus even if

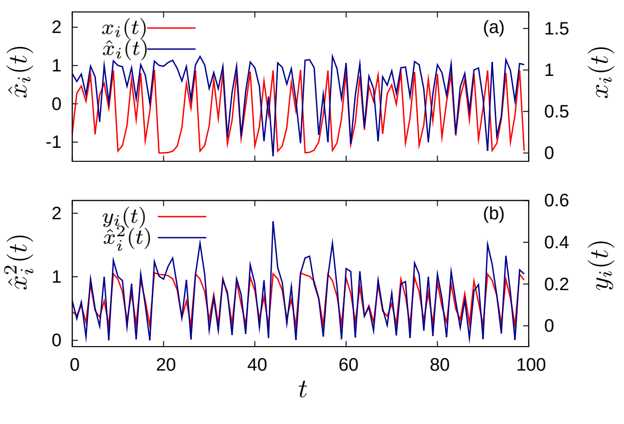

Figure 4: Example of an original time series and a reconstructed time series obtained from an evolving network (the logistic map was used for the dynamics). The original time series $x_i(t)$ (red line) and the reconstructed time series $\hat{x}_i(t)$ (dark blue line) are shown in (a). The transformed original time series $y_i(t)$ (red line) and the squared reconstructed time series $\hat{x}_i^2(t)$ by Eq.(8) (dark blue line) are shown in (b).

we obtain precise arrangements of the nodes, the signatures of $\hat{x}_i(t)$ might be inversed. Then, when we reconstruct the time series of the state of a node $x_i(t)$, the signature of a reconstructed state $\hat{x}_i(t)$ is often inversed. Figure 4 shows an example. In Fig.4(a), the original time series $x_i(t)$ (red line) and the reconstructed time series $\hat{x}_i(t)$ (dark blue line) are not similar because the signatures of the values of the states are partially inversed.

To evaluate the performance of our method without this effect of the inversion of signatures, we transformed $x_i(t)$ to a squared time series $y_i(t) = (x_i(t) - \bar{x}_i)^2$, where $\bar{x}_i = \frac{1}{T}\sum_{t=1}^{n} x_i(t)$. We then calculated the correlation coefficient between the transformed original time series $y_i(t)$ and the squared reconstructed time series $\hat{x}_i^2(t)$.

If we compare $\hat{x}_i^2(t)$ and $y_i(t)$, we can see high correspondences between them from Fig.4(b). In this case, the correlation coefficient between $y_i(t)$ and $\hat{x}_i^2(t)$ is about 0.94. This result indicates that the proposed method can reconstruct the time series of $x_i(t)$.

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