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Bifurcation Phenomena of Replicator Dynamics with Dynamic Capitation Tax

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Abstract—In the population of many selfish players, the purpose of each player often conflicts with the total purpose of the population. In such a situation, the "government" which has the comprehensive perspective is needed for governing the population. Recently, to model such a situation, replicator dynamics with capitation taxes has been proposed. However, the amount of the capitation tax is assumed to be constant and independent of the population state. In this paper, we propose a model that the government changes the capitation tax depending on the population state. Using a two-strategy game, we investigate the stability conditions and related bifurcations of equilibrium points of our model.

1. Introduction

In the population of many selfish players, the purpose of each player often conflicts with the total purpose of the population [1]. In such a situation, a "government" which has the comprehensive perspective is needed for governing the population. The government collects a tax from each player and reallocates it depending on a target state of the government. The tax is roughly classified into rate taxations and capitation taxes. In the former, the tax is determined based on payoffs the players earn while it is fixed in the latter. The authors proposed replicator dynamics with such taxes to analyze their effects on players' behaviors [2, 3]. In the previous work, the amount of taxes is independent of the population state.

In this paper, we deal with the capitation tax and consider the case that the government controls its amount depending on the population state. We regard the government as a game player. Its strategy is the amount of the tax and its payoff is a sum of benefits and a cost of its taxation. We extend replications dynamics proposed by Ref. [3] to describe the co-evolution of both the populations and the government. To investigate qualitative properties of such taxation, we focus on two-strategy game and investigate stability conditions of equilibrium points and related bifurcations.

2. Capitation Tax and Subsidy

Let P be the population of players. Suppose that $\Phi_p = \{1, 2, \cdots, m_p\}$ is a set of pure strategies of P, and S_p is a set of population states of P. A population state $s_p = (s_p^1, s_p^2, \cdots, s_p^{m_p})^T \in S_p$ is a distribution of strategies in the population P, where s_p^i is the proportion of players with a pure strategy $i \in \Phi_p$. Let $r_p^i : S_p \to \mathbb{R}$ be the payoff function for the players of P with the pure strategy $i \in \Phi_p$ and $\bar{r}_p(s_p)$ be the average payoff, i.e., $\bar{r}_p(s_p) = \sum_{i\in\Phi_p} s_p^i r_p^i(s_p)$. In this paper, we assume that the payoff function $r_p^i(s_p)$ is given by $r_p^i(s_p) = e_{m_p}^{iT} A s_p$ for simplicity, where e_l^i is the l-dimensional unit vector such that the ith element equals 1 and an $m_p \times m_p$ matrix A is called payoff matrix. Replicator dynamics which describes evolutions of distributions of strategies of P is given as follows [4]:

$$\dot{s}_p^i = s_p^i \left\{ r_p^i(s_p) - \bar{r}_p(s_p) \right\}. \tag{1}$$

We consider that the government collects capitation taxes from all players of the population P and reallocates them as subsidies depending on the desirable target population state. Then, the payoff function of players with the capitation taxes and the subsidies is given as follows [3]:

$$r_p^i(s_p) - \tau + \tau \frac{s_p^{i*}}{s_p^i} = r_p^i(s_p) - \tau \left(1 - \frac{s_p^{i*}}{s_p^i} \right), \tag{2}$$

where $\tau \geq 0$ is the amount of the capitation taxes and $s_p^* = (s_p^{1*}, \cdots, s_p^{m_p*})^T$ is the target population state. Note that, the average payoff of players $\bar{r}(s_p)$ is independent of the capitation tax τ since all collected taxes are assumed to be reallocated.

Substituting the right-hand side of Eq. (2) for players' payoff function $r_p(s_p)$ of Eq. (1), we have replicator dynamics with capitation taxes and subsidies as follows [3]:

$$\dot{s}_p^i = s_p^i \left\{ r_p^i(s_p) - \bar{r}_p(s_p) \right\} + \tau \left(s_p^{i*} - s_p^i \right). \tag{3}$$

Note that we allow $\tau > r_p^i(s_p)$ for all $i \in \Phi_p$. Moreover, suppose that every pure strategy is adopted by some players at least in the initial state, that is, $s_p^i(0) > 0$ is assumed to hold for any pure strategy $i \in \Phi_p$. By this assumption, within finite-time intervals, $s_p^i > 0$ holds for all $i \in \Phi_p$ and Eq. (2) is well-defined.

Equation (3) is given by adding the negative feedback term $\tau(s_p^{i*} - s_p^i)$ to the conventional replicator dynamics Eq. (1), and the tax τ is considered as a feedback gain.

It has been proved that the target state of Eq. (3) is also an equilibrium point if it is an equilibrium point of Eq. (1) and Eq. (3) is invariant under a *local shift* of the payoff matrix A, where the local shift is the addition of a constant to all elements of a column of A [3]. The stability condition of the target state of Eq. (3) has also been derived [3]. We focus on the case that the target state s_p^* is an equilibrium point of Eq. (1).

3. Dynamic Capitation Tax

Suppose that τ_{max} is the maximum capitation tax which the government can impose. The capitation tax $\tau \in [0, \tau_{max}]$ can be described by $\alpha \in [0, 1]$ as $\tau = \alpha\tau_{max}$. Suppose that $\Phi_g = \{1, 2\}$ and S_g are sets of pure and mixed strategies of the government, respectively. We call the first strategy *maximum taxation* and the second strategy *zero taxation*. The maximum taxation and the zero taxation mean that the government imposes τ_{max} and 0 as the capitation taxes τ, respectively. A mixed strategy $s_g = (\alpha, 1 - \alpha) \in S_g$ defines a capitation tax between those two strategies.

In general, the government's taxation takes a cost and its efficiencies depend on the current population state. Then, the government must be willing to adopt the most effective amount of taxes to the current population state. Thus, in this section, we consider the government changes its strategy based on its own payoffs.

Let $r_g^i : S_p \times S_g \to \mathbb{R}$ be a payoff function for the government with the pure strategy $i \in \Phi_g$ and $\bar{r}_g(s_p, s_g)$ be the government's current payoff, i.e., $\bar{r}_g(s_p, s_g) = \alpha r_g^1(s_p, s_g) + (1-\alpha) r_g^2(s_p, s_g)$. We suppose that the government increases the taxes by increasing α in proportion to differences between the payoffs of the maximum taxation strategy r_g^1 and the current payoffs $\bar{r}_g$ which the government earns. Then, the dynamics of α is also modeled by the following replicator dynamics:

$$\dot{\alpha} = \alpha(1 - \alpha)\left\{r_g^1(s_p, s_g) - r_g^2(s_p, s_g)\right\}. \tag{4}$$

For simplicity, we assume that the payoff function $r_g^i(s_p, s_g)$ is given by

$$r_g^i(s_p, s_g) = e_2^{iT} B s_p + e_2^{iT} C s_g, \tag{5}$$

where the payoff matrices B and C are given by

$$B = \begin{bmatrix} b_1 & \cdots & b_{m_p} \\ 0 & \cdots & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

We consider that the matrix B represents the government's benefit depending on the current population state of P and the matrix C represents a cost of the government's taxation depending on the current α. Therefore, the elements $b_1, \cdots, b_{m_p}$ are assumed to be nonnegative and c_1 is assumed to be negative. Moreover, we assume that the zero taxation strategy makes no benefit and cost. Then, all elements of the second lows of both matrices B and C are set to 0. By these assumptions, Eq. (4) is rewritten as follows:

$$\dot{\alpha} = -c_1\alpha(1 - \alpha)\left(\sum_{i\in\Phi_p} \beta_i s_p^i - \alpha\right), \tag{7}$$

where $\beta_i = -b_i/c_1$ for all $i \in \Phi_p$. Since β_i is the ratio of the government's benefits b_i to the taxation cost c_1, we consider $\sum_{i\in\Phi_p} \beta_i s_p^i$ as a cost-efficiency of the government's taxation at population state s_p.

4. Two-Strategy Game

In this section, to investigate stability of equilibrium points of our proposed replicator dynamics, we consider a two-strategy game. In the two-strategy game, we suppose that players of the population P have two strategies. Equation (3) is invariant under a local shift of payoff matrix A. Therefore, without loss of generality, we set the payoff matrix A to

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}. \tag{8}$$

Since $s_p^1 + s_p^2 = s_p^{1*} + s_p^{2*} = 1$, we have

$$\dot{s}_p^1 = \left(s_p^{1*} - s_p^1\right)\left\{d_1\left(s_p^1\right)^2 + d_2 s_p^1 + d_3\right\}, \tag{9}$$

$$\dot{\alpha} = -c_1\alpha(1 - \alpha)\left\{(\beta_1 - \beta_2) s_p^1 + \beta_2 - \alpha\right\}, \tag{10}$$

where

$$d_1 = a_1 + a_2, \tag{11}$$

$$d_2 = d_1\left(s_p^{1*} - 1\right) - a_2, \tag{12}$$

$$d_3 = d_2 s_p^{1*} + a_2 + \tau_{max}\alpha. \tag{13}$$

In the two-strategy game, two population states $s_p = (0, 1)^T$ and $(1, 0)^T$ are always equilibrium points of Eq. (1). Moreover, if $a_1 a_2 > 0$ holds, then $(\eta, 1 - \eta)^T$ is also an equilibrium point, where $\eta = a_2/(a_1 + a_2)$. Therefore, we consider three population states $(0, 1)^T$, $(1, 0)^T$, and $(\eta, 1 - \eta)^T$ as a target state s_p^*. However, we have the game with target state $s_p^* = (0, 1)^T$ by swapping pure strategies 1 and 2 of the game with $s_p^* = (1, 0)^T$. So, as a target population state, we select the following two equilibrium points: $s_p^* = (1, 0)^T$ on the boundary of S_p and $s_p^* = (\eta, 1 - \eta)^T$ in the interior of S_p.

We have $\dot{s}_p^1 = 0$ if $s_p^1 = s_p^{1*}$ or $d_1\left(s_p^1\right)^2 + d_2 s_p^1 + d_3 = 0$. On the other hand, we have $\dot{\alpha} = 0$ if $\alpha = 0, 1$, or $(\beta_1 - \beta_2)s_p^1 + \beta_2 - \alpha = 0$. Thus, $\dot{s}_p^1 = 0$ holds on the curve

$$l_1 : \alpha = -\frac{1}{\tau_{max}}\left\{d_1\left(s_p^1\right)^2 + d_2 s_p^1 + \left(d_2 s_p^{1*} + a_2\right)\right\}. \tag{14}$$

$\dot{\alpha} = 0$ holds on the following line

$$l_2 : \alpha = (\beta_1 - \beta_2) s_p^1 + \beta_2 := \alpha_{l_2}\left(s_p^1\right). \tag{15}$$

Table 1 shows equilibrium points of Eqs. (9) and (10), and their existence conditions. Since the points W_l and W_r are intersections of l_1 and l_2, their s_p^1-coordinates ξ_l and ξ_r satisfy

$$d_1\xi^2 + w_1\xi + w_2 = 0, \tag{16}$$

Table 1: Equilibrium points of Eqs. (9) and (10), and their existence conditions.

Equilibrium Point (s_p^1, α)	Existence conditions
T_0 $(s_p^{1*}, 0)$, T_1 $(s_p^{1*}, 1)$	always
T_{l_2} $(s_p^{1*}, \alpha_{l_2}(s_p^{1*}))$	$\left(1 - s_p^{1*}\right)\beta_2 \le 1 - s_p^{1*}\beta_1$
W_l $(\xi_l, \alpha_{l_2}(\xi_l))$ W_r $(\xi_r, \alpha_{l_2}(\xi_r))$	$(w_1)^2 \ge 4d_1 w_2$ $\xi, \alpha_{l_2}(\xi) \in [0, 1]$ (for $\xi = \xi_l, \xi_r$)
$O(0, 0)$,	always
$V(\eta, 0)$	$s_p^* = (1, 0)^T$ and $a_1 a_2 > 0$
$V(1, 0)$	$s_p^* = (\eta, 1 - \eta)^T$

where

$$w_1 = d_2 + \tau_{max}(\beta_1 - \beta_2), \tag{17}$$

$$w_2 = d_2 s_p^{1*} + a_2 + \tau_{max}\beta_2, \tag{18}$$

and $\xi_l \le \xi_r$. Their α-coordinates are given by Eq. (15). The target population state s_p^* corresponds to the line T_0T_1, that is, any point on the line is a target state. Therefore, the achievement of the target state requires that the point T_{l_2} or T_1 is asymptotically stable.

By the linearization of Eqs. (9) and (10) around each equilibrium point, we can investigate its stability conditions and related bifurcations which depend on the parameters β_1, β_2, and τ_{max}.

We define ζ as follows:

$$\zeta = -3d_1\left(s_p^{1*}\right)^2 + 2(d_1 + a_2)s_p^{1*} - a_2. \tag{19}$$

If $\tau_{max} < \zeta$ holds, then the equilibrium points T_0, T_{l_2}, and T_1 are unstable independent of the parameters β_1 and β_2, and the target state cannot be achieved. Therefore, we consider the case of $\tau_{max} > \zeta$ in this paper.

We have a bifurcation set of Eqs. (9) and (10) in the β_1-β_2 plane under $\tau_{max} > \zeta$ as shown in Fig. 1. The boundaries of the regions in Fig. 1 are given as follows:

$$\mathrm{Tc}_1 : \beta_1 = -\frac{1 - s_p^{1*}}{s_p^{1*}}\beta_2 + \frac{\zeta}{\tau_{max}s_p^{1*}}, \tag{20}$$

$$\mathrm{Tc}_2 : \beta_1 = -\frac{1 - s_p^{1*}}{s_p^{1*}}\beta_2 + \frac{1}{s_p^{1*}}, \tag{21}$$

$$\mathrm{SN} : \beta_1 = \beta_2 + \frac{-d_2 - 2\sqrt{d_1\tau_{max}\beta_2}}{\tau_{max}} \quad \text{for} \quad \beta_2 < \frac{d_1}{\tau_{max}}. \tag{22}$$

Note that we assume that $a_1 < 0$ and $a_2 > \tau_{max}$ in Fig. 1(a). These conditions imply that $d_1 > 0$. If $a_1 > 0$ (*resp.* $a_2 < \tau_{max}$) holds, then Region I (*resp.* II_l) does not exist. If $d_1 < 0$, then not only Region II_l but also III_l does not exist. Also note that we assume that $a_1 > \tau_{max}$ and $a_2 > \tau_{max}$ hold in Fig. 1(b). If $0 < a_1 < \tau_{max}$ (*resp.* $0 < a_2 < \tau_{max}$) holds, then Region III_r (*resp.* III_l) does not exist. If $a_1 < 0$ and $a_2 < 0$, not only Regions III_r and III_l but also Regions II_r, II_l, and I do not exist.

Figures 2 and 3 show examples of their phase portraits in each region of Figs. 1(a) and 1(b), where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tau_{max} = 2, \tag{23}$$

Table 2: Asymptotically stable and unstable equilibrium points of Eqs. (9) and (10) in each region of Fig. 1(b).

	Stable points	Unstable points
Region I	W_l, W_r	O, V, T_0, T_1, T_{l_2}
Region II_l	W_l, T_{l_2}	O, V, W_r, T_0, T_1
Region II_r	W_r, T_{l_2}	O, V, W_l, T_0, T_1
Region III_l	W_l, T_1	O, V, W_r, T_0, T_{l_2}
Region III_r	W_r, T_1	O, V, W_l, T_0, T_{l_2}
Region IV	T_{l_2}	O, V, T_0, T_1
Region V	T_1	O, V, T_0, T_{l_2}

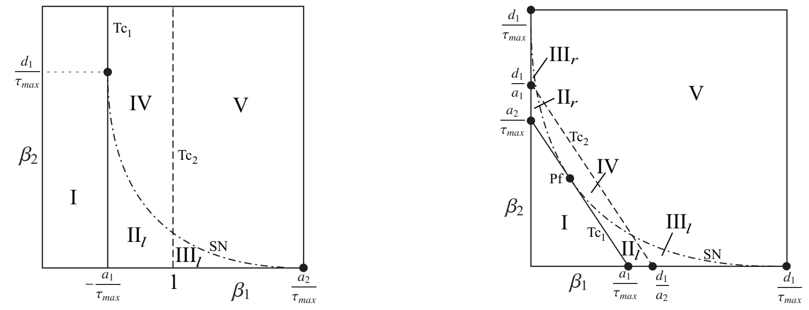
and

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tau_{max} = 3, \tag{24}$$

respectively. Black squares and points correspond to stable and unstable equilibrium points, respectively. Note that $\beta_1 = b_1$ and $\beta_2 = b_2$ hold in these cases.

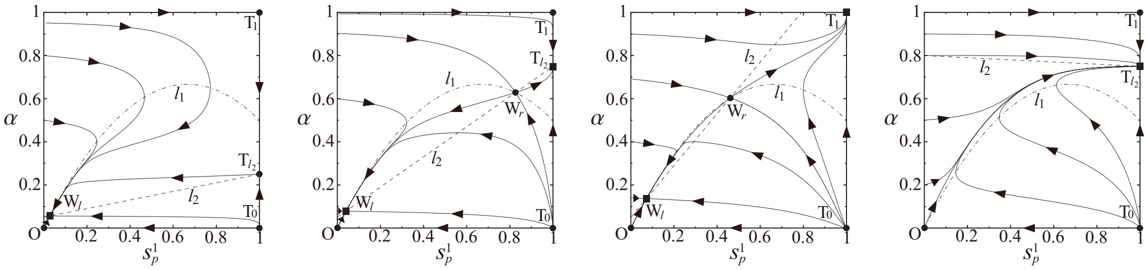
In Region I, W_l and W_r (if $s_p^* = (\eta, 1-\eta)^T$) are asymptotically stable equilibrium points as shown in Figs. 2(a) and 3(a). On the boundary Tc_1, a transcritical bifurcation occurs, and the stability of T_{l_2}, and W_r or W_l is exchanged in Region II as shown in Figs. 2(b) and 3(b). On the boundary Tc_2, a transcritical bifurcation which results from the collision of T_{l_2} and T_1 occurs and their stability is exchanged in Region III as shown in Figs. 2(c) and 3(c). On the boundary SN, a saddle-node bifurcation occurs, and W_l and W_r collide and disappear in Regions IV and V as shown in Figs. 2(d) and 3(d). At the contact point Pf of Regions I and IV, three equilibrium points T_{l_2}, W_l, and W_r collide, and a pitchfork bifurcation occurs. W_l and W_r disappear and T_{l_2} is stabilized at the same time. We summarize stability conditions of each equilibrium point of Eqs. (9) and (10) in each region of Fig. 1 as shown in Table 2.

From definition of β, the value of $(\beta_1 - \beta_2)s_p^1 + \beta_2$ describes the cost-efficiency of the government's taxation at a population state $s_p = (s_p^1, 1 - s_p^1)^T$. By changing the parameters β_1 and β_2, the cost-efficiency also changes. In Region I, the cost-efficiency of the taxation at the target state s_p^* is too low and the government cannot impose a sufficiently large amount of tax on the players. Then, the target population state cannot be achieved. In Regions II and III, the cost-efficiency of the taxation at s_p^* becomes high enough to achieve the target state. In these regions, however, since the cost-efficiencies at some population states $s_p \in S_p$ are not so high, the government must select a sufficiently large value as the initial tax for the achievement of the target state. In Regions IV and V, since the cost-efficiencies at not only the target state but also any other population state are sufficiently high, the government can impose a sufficiently large amount of tax at any population state and control the population to the target state for any initial population state and any initial tax.



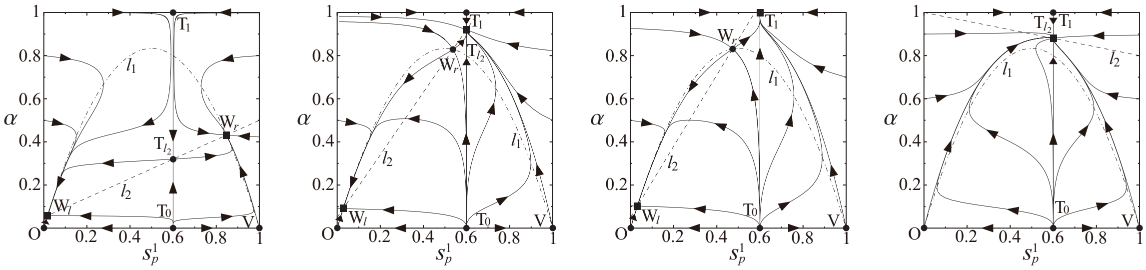
(a) The target state $s_p^* = (1, 0)^T$ (b) The target state $s_p^* = (\eta, 1-\eta)^T$.

Figure 1: Bifurcation set of Eqs. (9) and (10). Denoted by Tc_1 and Tc_2 correspond to transcritical bifurcations, SN corresponds to a saddle-node bifurcation, and Pf corresponds to a pitchfork bifurcation.



(a) Region I (b) Region II_l (c) Region III_l (d) Region IV

Figure 2: Examples of phase portraits where the payoff matrices are defined by Eq. (23) and $\tau_{max} = 2$ in each region of Fig. 1(a). Black squares are stable equilibrium points and black points are unstable equilibrium points.



(a) Region I (b) Region II_l (c) Region III_l (d) Region IV

Figure 3: Examples of phase portraits where the payoff matrices are defined by Eq. (24) and $\tau_{max} = 3$ in each region of Fig. 1(b).

5. Conclusions

In this paper, we have considered the government as a game player and extended the model in [3]. We have defined the government's strategy as the maximum taxation and the zero taxation and its payoff as a sum of benefits and a cost of the government's taxation. Moreover, we have proposed replicator dynamics which describes the co-evolution of strategies of populations and the government and, using a two-strategy game, have investigated the stability conditions and related bifurcations of equilibrium points.

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