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On Some Difference Between Chua's Circuits with Different Nonlinearities

Mieczyslaw Jessa†

† Faculty of Electronics and Telecommunications, Poznan University of Technology,
ul. Polanka 3, 60-965 Poznan, Poland
Email: mjessa@et.put.poznan.pl

Abstract – In this paper we show that Chua's circuit with nonlinearity being an odd function has a property which does not occur in the same circuit when the nonlinearity is not an odd function. We discuss the influence of this property on application of the circuit as a source of signal, mimic noise generated by a nondeterministic physical system.

1. Introduction

During the last few decades, the chaotic dynamics of many continuous systems have been evaluated in detail. One of these systems is Chua's circuit. An idealized Chua's circuit is shown in Figure 1 [2]-[4], [6], [9], [10], [14], [15]. It contains simple electronic components like resistors *R*, capacitors *C*, inductors *L* and a nonlinear, memory-less element R_N . Energy is pumped into the circuit via operational amplifiers used in the real circuit.

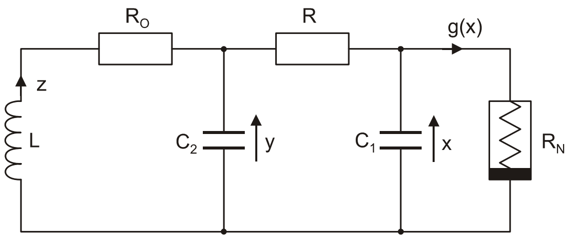

Fig. 1. An idelized Chua's circuit.

The dynamics of Chua's circuit is modeled by the set of ordinary differential equations

$$\begin{aligned}\dot{x} &= \frac{1}{RC_1}\left(y-x\right)-\frac{1}{C_1}g(x)\\ \dot{y} &= \frac{1}{RC_2}(x-y)+\frac{1}{C_2}z \quad ,\\ \dot{z} &= -\frac{1}{L}y-\frac{R_0}{y}z\end{aligned} \qquad (1)$$

where $R, C_1, C_2, L \neq 0$, and $g(x)$ is a nonlinear function. In this simple setup, it is possible to observe experimentally almost all phenomena characteristic for nonlinear dynamical systems, which gives Chua's circuit a significant advantage over other chaotic systems.

In the paper, we consider the difference between Chua's circuit with an odd $g(x)$, and the same circuit with $g(x)$ not being an odd function. The function $g(x)$ is odd if and only if $g(x) = -g(-x)$.

2. Non-Achievable Points of Non-Periodic Orbits

Chua's circuit is an example of a dynamical system. Generally, a dynamical system can be defined in many ways. In this paper, we use the following definition [7].

Definition 1
A dynamical system $\left(T^{\alpha}, X, \mu\right)$ on metric space X with measure μ is a family of transformations $T^{\alpha}: X \to X$, satisfying the following conditions:

(a) $T^0(x) = x$ for all $x \in X$;

(b) $T^{\alpha}(T^{\alpha'}(x)) = T^{\alpha+\alpha'}(x)$ for all $x \in X$,

where the parameter α (time) may be continuous or discrete.

Denoting the set of values of parameter α as Γ , we can define the following cases: $\Gamma = R$, $\Gamma = R_+$, $\Gamma = Z$, $\Gamma = Z_+$. If $\Gamma = R$ or $\Gamma = R_+$, a continuous-time dynamical system is obtained. For $\Gamma = R$ the system is reversible and for $\Gamma = R_+$ it is not reversible, that is, it is defined only for $\alpha \geq 0$. Similarly, if $\Gamma = Z$ or $\Gamma = Z_+$, a discrete-time dynamical system is obtained. For $\Gamma = Z$ the system is reversible and for $\Gamma = Z_+$ it is not reversible. If $X \subseteq R^d, d < \infty$, we obtain a *d*-dimensional continuous-time or discrete-time dynamical system. The function $T^{\alpha}(x_0), x_0 \in X$, considered as a function of α , is called an orbit or a trajectory of the dynamical system $\left(T^{\alpha}, X, \mu\right)$. The orbit of the system is denoted as $\{x_{\alpha}\}$. To distinguish a continuous-time dynamical system from a discrete-time dynamical system, the latter one is denoted as $\left(T^n, X\right)$, where $n \in Z$ and the former one as $\left(T^t, X\right)$. The orbit of system $\left(T^t, X\right)$ is written as $\{x_t\}$ and the orbit of system $\left(T^n, X\right)$ as $\{x_n\}$.

The key to understanding the difference between Chua's circuits with different $g(x)$ is the definition of odd-symmetric dynamical system.

Definition 2

A dynamical system $\left(T^{\alpha}, X, \mu\right)$ is said to be odd-symmetric under the transformation $x \rightarrow -x$ of state variables $(x^1, x^2, .., x^d) \rightarrow (-x^1, -x^2, .., -x^d)$ if

$$T^{\alpha}(x) = -T^{\alpha}(-x). \tag{2}$$

It is known that the evolution of a chaotic system is sensitive to the initial conditions and the values of the parameters: any small change in an initial condition or parameter, changes the future evolution of the system dramatically. Generally, knowing the previous states of a chaotic system, we cannot precisely determine which states cannot be achieved after some time. Exceptions are the states already generated, which cannot be repeated during a non-periodic evolution of a chaotic system. The difference between Chua's circuits with different $g(x)$ results from the following new theorem:

Theorem

Points x and $-x$ are not elements of the same non-periodic orbit of an odd-symmetric dynamical system $\left(T^{\alpha}, X, \mu\right)$.

Proof

Let us assume that a non-periodic orbit of an odd-symmetric dynamical system $\left(T^{\alpha}, X, \mu\right)$ contains both x and $-x$, where $x, -x \in X$. At time instant α_1 the orbit starting at a certain $x_0 \in X$ goes through a point, e.g. $x \in X$. If point $-x \in X$ was not obtained for $\alpha < \alpha_1$ and it ought to belong to the same non-periodic orbit, then it must be obtained after time α_2, starting from instant α_1. In other words, it should be true that

$$T^{\alpha 2}\left(T^{\alpha 1}(x_0)\right) = -x. \tag{3}$$

Since the orbit goes through $x \in X$, it is also true that

$$T^{\alpha 1}(x_0) = x. \tag{4}$$

From (3) and (4)

$$T^{\alpha 2}\left(x\right) = -x. \tag{5}$$

After time α_2, we obtain

$$T^{\alpha 2}\left(T^{\alpha 2}(x)\right) = T^{\alpha 2}(-x). \tag{6}$$

If $\left(T^{\alpha}, X, \mu\right)$ is odd-symmetric

$$T^{\alpha_2}(-x) = -T^{\alpha_2}(x). \tag{7}$$

Substituting (5) into (7) we obtain

$$T^{\alpha_2}\left(-x\right) = x. \tag{8}$$

If $\left(T^{\alpha}, X, \mu\right)$ is odd-symmetric, we first obtain x, then $-x$, and finally x, i.e. a periodic orbit. This contradicts the assumption that the orbit is non-periodic. Thus, if x is an element of a non-periodic orbit, then $-x$ cannot be an element of the same orbit and vice versa, which ends the proof.

If $g(x)$ is odd, equations (1) are symmetric under the transformation $(x, y, z) \rightarrow (-x, -y, -z)$ and the dynamical system is odd-symmetric. Orbits starting from (x_0, y_0, z_0) and from $(-x_0, -y_0, -z_0)$ are conjugate, and the points (x, y, z) and $(-x, -y, -z)$ cannot belong to the same non-periodic orbit. However, they can belong to the same periodic orbit. Chaotic orbits with these properties can be obtained for a smooth cubic nonlinearity [14]:

$$g(x) = ax + bx^3. \tag{9}$$

The existence of points that cannot be elements of a non-periodic orbit, unequivocally related to points previously generated, is a new property of dynamical systems. This property is characteristic for Chua's circuits with odd $g(x)$ and for a relatively small number of other systems. It is not observed in most continuous-time chaotic systems described in the literature because these systems are not odd-symmetric [12]. Examples of systems that are not odd-symmetric include, e.g., the Lorenz system, Chen's system or the Rössler system.

The differential equations describing the dynamics of the Lorenz system have the following form [8]

$$\begin{aligned} \dot{x} &= 10\left(y - x\right) \\ \dot{y} &= ax - y - xz, \\ \dot{z} &= xy - \frac{8}{3}z \end{aligned} \tag{10}$$

where $a \neq 0$. The set of equations (10) is symmetric under the transformation $(x, y, z) \rightarrow (-x, -y, z)$. Equations describing Chen's system [1]

$$\begin{aligned} \dot{x} &= a\left(y - x\right) \\ \dot{y} &= rx + cy - xz, \\ \dot{z} &= xy - bz \end{aligned} \tag{11}$$

where $a, b, c, r \neq 0$, are symmetric under the same transformation as for the Lorenz system. In contrast with the Lorenz attractor, Chen's attractor is topologically more complex [12]. An example of a dynamical system with chaotic orbits, which is not symmetric for any coordinate, is the Rössler system [11]:

$$\begin{aligned} \dot{x} &= -\left(y + z\right) \\ \dot{y} &= x + ay \quad, \\ \dot{z} &= b + xz - cz \end{aligned} \tag{12}$$

where $a, b, c \neq 0$. As was the case above, apart from points previously generated we cannot precisely specify points that cannot belong to this non-periodic trajectory.

An example of chaotic system with the same property as Chua's circuit with odd $g(x)$ is Thomas' system [13]

$$\begin{aligned} \dot{x} &= -bx + \sin y \\ \dot{y} &= -by + \sin z \quad, \\ \dot{z} &= -bz + \sin x \end{aligned} \tag{13}$$

where $b > 0$. Another example is the double scroll system [5]

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -a(z + y + x - \operatorname{sgn} x) \end{aligned} , \qquad (14)$$

where $a > 0$. Both systems are symmetric under the transformation $(x, y, z) \to (-x, -y, -z)$.

The theorem proved above shows that odd-symmetric continuous-time chaotic systems may generate signals that are theoretically less noise-like than signals produced by continuous-time systems that are not odd-symmetric. Theoretically, we should avoid using such systems as a source of noise-like signals, e.g. in generating random numbers. Using chaotic signals for random number generation is an important application of chaotic systems. Since the same theorem can be easily extended to odd-symmetric and even-symmetric discrete-time chaotic systems, these systems should also not be used as a sources of noise-like signals. On the other hand, a real circuit is not free from nondeterministic noise, which in practice eliminates this disadvantage. However, chaotic signals generated by an isolated odd-symmetric continuous-dynamical system and by a system that does not have this property cannot be considered as signals with exactly the same properties, even when these systems have identical values of known parameters such as Lyapunov exponents, KS entropy, dimensions, etc.

3. Conclusion

The main point of this paper was the difference between Chua's circuits with different nonlinearities. A theorem was proved that shows that non-periodic orbits of an odd-symmetric dynamical system have a property that does not occur for non-periodic orbits generated by a dynamical system that is not odd-symmetric. This property is independent of the parameters describing the chaotic dynamics of the system. It introduces an additional difference between chaotic signal and noise coming from a nondeterministic physical source. This difference is not observed for continuous-time chaotic systems that are not odd-symmetric. Chua's circuit is probably the only circuit capable of demonstrating in the same hardware the behavior characteristic for both odd-symmetric systems and systems lacking this symmetry.

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