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A Spectral Reconstitution Method of Potential Function in Dynamical System

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Abstract—Phase structure of dynamical system governs their trajectories, which depend on the initial condition. Accordingly, the power spectra of trajectories include the structural information of the phase space of the nonlinear system. In this paper, the potential function of a low-dimensional nonlinear system is estimated through the power spectra of the trajectories from the given initial conditions. It is clarified that a method has a potential to reconstitute the potential function in the dynamical system.

1. Introduction

Nonlinear systems show inherently a variety of phenomenon including chaos [1, 2, 3]. Most of analyzing methods to grasp nonlinear systems are based on the temporal characteristic in phase space: Lyapunov exponent, Poincaré map, correlation function, invariant measure, statistics value, and so forth [2, 3, 4, 5]. In particular, the analyses based on operators are proposed for extracting invariant measures and statistics values [6, 7, 8]. The analysis for Arnold web resonances are proposed to an experimentally accessible atomic system that may well form a paradigm for phenomena in several degrees of freedom [9]. The method fundamentally focuses on power spectrum distribution in parameter space [10]. The power spectrum has the information of trajectories which are governed by the phase structure and the realm of solutions.

In this paper, the recurrent dynamics is focused on through power spectrum for the analysis of low-dimensional nonlinear dynamical system. In addition, a spectral reconstitution method is proposed for extracting potential function based on a spectral decomposition of dynamical system.

2. Brief Summary of Power Spectrum and System

2.1. Power Spectrum of Trajectory

Power spectrum describes the power distribution in angular frequency ω [11]. In the numerical analysis, every trajectory is obtained as a time series for integration step. Here, an autocorrelation function of the time series is applied to derive the power spectrum of the time series. The autocorrelation function is an integral function on a time shift, which is a measure of correlation defined by [11, 12]:

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f(\tau+t) f(\tau) \mathrm{d}t. \qquad (1)$$

$f(t)$ denotes a trajectory in dynamics. Fourier transform of the autocorrelation function gives a power spectrum:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) \mathrm{e}^{-\mathrm{i}\omega\tau} \mathrm{d}\tau. \qquad (2)$$

In the following section, phase structure is discussed through the power spectrum decomposition for nonlinear dynamical systems with relation to initial conditions.

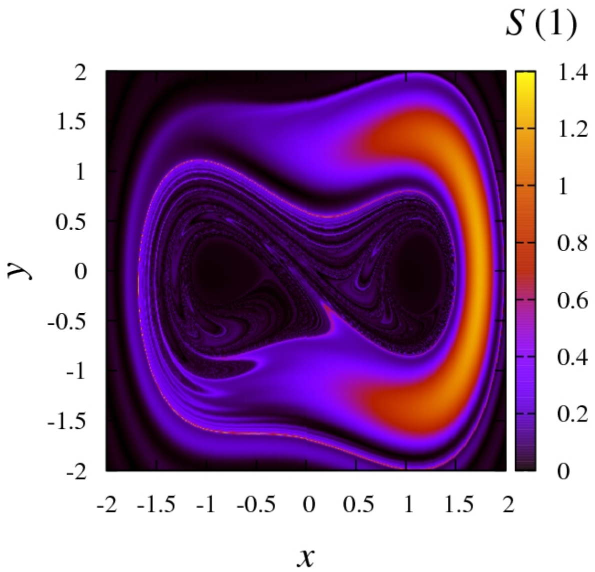


Figure 1: Fundamental harmonic power spectrum distribution in initial value phase space. $k = 0$ and $B = 0.1$ in Eq. (3). Color shows power level.

2.2. Power Spectrum Distribution in Initial Value Space

Here, a Duffing type nonlinear dynamical system is picked up for discussion:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -ky - x^3 + x + B\cos t, \end{cases} \tag{3}$$

where $-ky$ denotes damping term and $B\cos t$ the external force term. First, the initial value space is split in lattice at 400×400. Secondly, the solution is calculated for each initial condition by the fourth order Runge-Kutta method until six cycles of the external force. Then, a power spectrum of the solution is estimated by foregoing method. Fig. 1 shows the fundamental harmonic power spectrum distribution in initial value space with $k = 0$ and $B = 0.1$. The result represents the information of trajectories which are governed by the phase space structure. In the later sections, we focus on the relationship between power spectrum and potential function.

3. Potential Function of Low Dimensional System

This section discusses potential function for the given low dimensional system and their distinction by spectral reconstitution. Here, the system is generally described as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= -\varepsilon ky - \dfrac{\partial U(x)}{\partial x} + \varepsilon B\cos t, \end{cases} \tag{4}$$

where $-\varepsilon ky$ denotes damping term and $\varepsilon B\cos t$ the external force term. ε is relatively small enough. After neglecting these terms, Eq. (4) becomes

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\partial U(x)}{\partial x}. \tag{5}$$

3.1. Monomial Potential

Here, we assumed a monomial potential function. The potential function is given as $a_n x^n$. a_n is a constant coefficient. It is also assumed that all trajectories are bounded. Then, $a_n > 0$ and n is even. By the symmetry of the potential function, the period T is equivalent to the term given by x_0. x_0 is a point where potential value shows peak. The relationship between the period T and the fundamental angular frequency ω holds $\omega = 2\pi/T$. Then, ω is represented by [13, 14] :

$$\omega = \sqrt{\frac{a_n\pi}{2}}\, n \frac{\Gamma(\frac{1}{n}+\frac{1}{2})}{\Gamma(\frac{1}{n})} |x_0|^{\frac{n}{2}-1}, \tag{6}$$

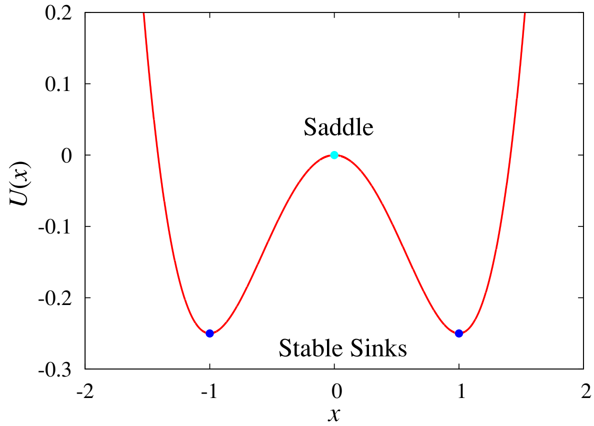


Figure 2: Potential function of $U(x) = x^4/4 - x^2/2$. Saddle exists at $x = 0$ and two stable sinks exist at $x = \pm 1$.

where $\Gamma(x)$ is Gamma function, which is defined as $\int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$.

3.2. Polynomial Potential

In this section, polynomial potential function is considered. Then, the potential function is generally given by

$$U(x) = \sum_{i=0}^{n} a_i x^i. \tag{7}$$

Assuming that all trajectories in the system are bounded, $a_n > 0$ and n is even. The lowest dimensional polynomial function possesses two wells under fourth potential function. That is, the potential function is depicted by $U(x) = x^4/4 - x^2/2$ as drawn in Fig. 2. The equilibrium points exist at $x = 0$ and $x = \pm 1$ for the potential function. One of them is a saddle at $x = 0$ and others are stable sinks at $x = \pm 1$.

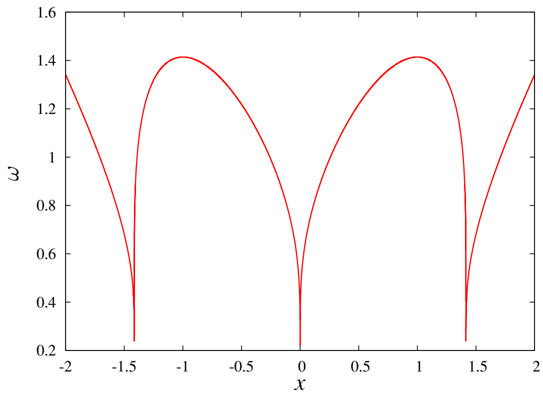
4. Spectral Estimation of Potential Function

In this section, the potential function is numerically estimated through power spectrum for the given potential function $U(x) = x^4/4 - x^2/2$. The following sections discuss the distinction of the potential function to reconstitute.

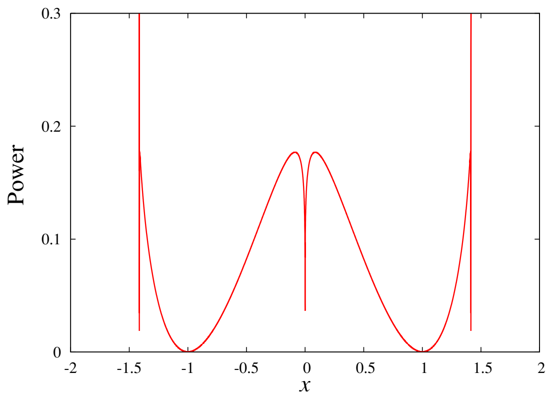
4.1. Decomposition of Power Spectrum

Decomposition of power spectrum from trajectories is expounded so that potential functions is reconstituted. Each trajectory is decomposed to power spectrum by the method in Section 2.1. The fundamental harmonic wave shows the main power of the trajectory.

Figure 3 shows the numerically obtained distributions along x. Their initial condition is set at $x(0) = x$,



(a) fundamental angular frequency ω



(b) power of fundamental angular frequency ω

Figure 3: Fundamental angular frequency ω and power of fundamental angular frequency ω under $U(x) = x^4/4 - x^2/2$. Their initial condition is set at $x(0) = x$, $\mathrm{d}x/\mathrm{d}t|_{t=0} = 0$.

$\mathrm{d}x/\mathrm{d}t|_{t=0} = 0$. They present the fundamental angular frequency ω and the power of all oscillations.

4.2. In Vicinity of Saddle

Here, the power spectrum is analyzed in the vicinity of a saddle at $x = 0$. The nearer the initial value x approaches the saddle, the lower the fundamental angular frequency ω becomes in Fig. 3(a). Fig. 4 shows the relationship between the period T and initial condition x. The eigen values of the saddle are obtained as $\lambda = \lambda_1, \lambda_2$, where $\lambda_1 = 1$, $\lambda_2 = -1$ at $x = 0$ for the potential function $U(x) = x^4/4 - x^2/2$. In the vicinity of the saddle, a trajectory is governed by the eigen vector. Here, we assume that the trajectory passes from x_1 to x_2. x_1 is an initial value in the vicinity of the saddle and x_2 is a point where $U(x_1) = U(x_2)$. Then the period T follows:

$$\frac{\lambda_1 T}{2} = \ln x_2 - \ln x_1 + \alpha, \tag{8}$$

where α denotes a linear error term. Fig. 4 also shows the estimation of the period T. Hence, the period T

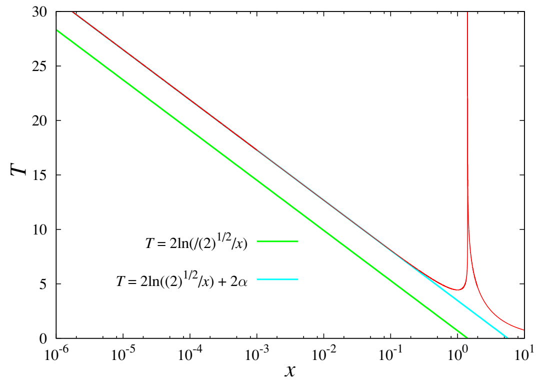


Figure 4: Relationship between the period T and initial condition x in the vicinity of saddle equilibrium point, under $U(x) = x^4/4 - x^2/2$. Aqua and green lines express Eq. (8), where $\alpha = 1.3863$ and $x_2 = \sqrt{2}$.

gives the the eigen value of a saddle in an unknown potential function $U(x)$. In the vicinity of the initial condition which has same potential value as the saddle, similar relationship the period T and the eigen value of a saddle is confirmed in Fig. 3.

4.3. In Vicinity of Stable Sinks

We discuss the power spectrum in the vicinity of stable sinks at $x = \pm 1$. For the stable sinks, the power spectral distribution of the fundamental angular frequency ω almost becomes 0 in Fig. 3(b). However, the fundamental angular frequency ω converges to 1.4 in Fig. 3(a). The result is depicted as follows:

$$\begin{aligned} U(x') &= \frac{1}{4}(x'+1)^4 - \frac{1}{2}(x'+1)^2 \\ &= \frac{1}{4}x'^4 + x'^3 + x'^2 - \frac{1}{4}, \end{aligned} \tag{9}$$

where $x' = x - 1$, which is a variational displacement around a center at $x = 1$. The potential function $U(x')$ has a second order term x'^2. Assuming sufficiently small x', the second order term x'^2 is predominant in the potential function $U(x')$. Therefore the potential function $U(x')$ can be approximated as $U(x') = x'^2$. The fundamental angular frequency ω becomes as $\sqrt{2}$ in Fig. 3(a). A fundamental angular frequency ω approximately gives a potential function $U(x)$ in a vicinity of stable sink.

4.4. Phase Structure in Large Area

It is important that a phase structure of the dynamical system is discussed in larger initial space. Fig. 5 shows one of the relationship between the fundamental angular frequency ω and the initial condition x. The fundamental angular frequency ω is proportional to the $|x|$ in Fig. 5. This is because the highest order

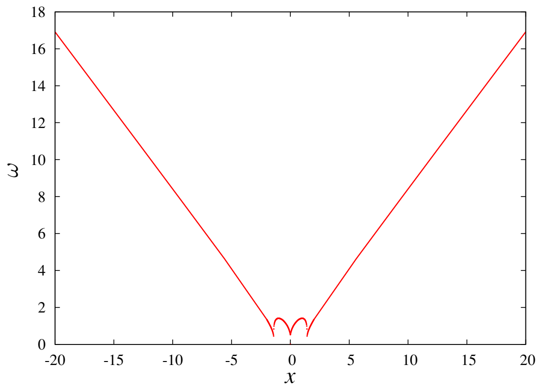


Figure 5: Relationship between fundamental angular frequency ω and initial condition x in large area. In the estimation, $U(x) = x^4/4 - x^2/2$.

term of the potential function is dominant according to the increase of amplitude. In Eq. (6) the fundamental angular frequency ω is proportional to the initial condition $|x|$ with the potential function of $U(x) = x^4/4$. The highest order term naturally appears.

4.5. Reconstitution of Potential Function

In the previous sections, vicinities of equilibrium points have characteristics. In particular, the period T in vicinity of saddle sets the eigen value of the saddle, and the period T in vicinity of stable sink represents a second order term of the potential function $U(x)$. Therefore, the potential function is approximately described as a quadratic potential function in vicinities of the equilibrium points. In addition, phase structure in larger area indicates max dimension n and the coefficient a_n of potential function. The spectral decomposition of dynamical system extracts a compartment of the potential function.

5. Summary

This paper focused on the relationship between power spectrum and potential function based on the fundamental harmonic power spectrum distribution. In vicinity of saddle, the trajectories had the information of the eigen values of the saddle. In addition, the potential function is approximately described as quadratic potential function in vicinities of the stable sinks. It is also confirmed that the highest order term of the potential function appeared in larger area.

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References

[1] Y. Ueda, "Steady Motions Exhibited by Duffing's Equation –A Picture Book of Regular and Chaotic Motions–", *in new approaches to nonlinear, problems in dynamics* (ed. PJ Holmes), pp. 311–322, 1980.

[2] Y. Oono and M. Osikawa, "Chaos in Nonlinear Difference Equation. I", *Prog. Theor. Phys.*, vol. 64 (1), pp. 54–67, 1980.

[3] J. M. T. Thompson and H. B. Stewart, "Nonlinear Dynamics and Chaos", chap. 1, (John Wiley & Sons, 1986).

[4] S. Wiggins, "Introduction to Applied Nonlinear Dynamical Systems and Chaos", chap. 10, chap. 29, (Springer, 2nd Ed., 2003).

[5] P. Bergé, Y. Pomeau, and C. Vidal, "Order within chaos–Towards a deterministic approach to turbulence–" , chap. 4, (John Wiley & Sons, 1986).

[6] J. Ding, "The Point Spectrum of Frobenius-Perrono and Koopman Operators", *P. Am. Math. Soc.*, vol. 126 (5), pp. 1355–1361, 1998.

[7] A. Lasota and M. C. Mackey, "Chaos, Fractals, and Nosise –Stochastic Aspects of Dynamics–", chap. 3, (Springer, 2nd Ed., 1998).

[8] I. Mezić, "Spectral Properties of Dynamical Systems, Model Reductionand Decompositions", *Nonlinear Dynam.*, vol. 41, pp. 309–325, 2005.

[9] J. V. Milczewski, G. H. F. Diercksen, and T. Uzer, "Computation of the Arnold's Web for Hydrogen Atom in Crossed Electric and Magnetic Fields", *Phy. Rev. Lett.*, vol. 76 (16), pp. 2890–2893, 1996.

[10] M. S. Bartlett, "Periodogram Analysis and Continuous Spectra", *Biometrika*, vol. 37 , pp. 1–16, 1950.

[11] E. J. Hannan, "Time Series Analysis", chap. 3 (Chapman & Hall/CRC, 1967).

[12] A. Papoulis, "The Fourier Integral and Its Applications", pp. 240–252, (McGraw-Hill, 1962).

[13] L. D. Landau and E. M. Lifshiz (Translated from Russian by J. B. Sykes and J. S. Bell), "Mechanics", pp. 25–29, (Pergamon Press, 3rd Ed., 1976).

[14] H. Goldstein, "Classical Mechanics", pp. 59–64, (Addison-Wesley, 2nd Ed., 1980).