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Development of nonlinear analysis tools on FPGA

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Abstract—Many mathematical models in various scientific fields are represented by complicated differential equations and such models cannot be solved analytically. Therefore, numerical analysis using computers is essential. However, when the system is more complicated, the analysis time is longer. In this study, we propose a fast numerical analysis method using FPGA (Field Programmable Gate Array). FPGA is a reconfigurable integrated circuit, and it is good at parallel processing. We develop the calculators of bifurcation diagrams and basins of attraction on the FPGA board. As a result, we obtain at most 22 times faster calculator than using CPU.

1. Introduction

Mathematical models are important to capture the essence of complicated natural phenomena. We consider that one of the most important mathematical models is the Hodgkin-Huxley neuron model [1]. Since the publication of this model, many conductance-based neuron models have been proposed. Among them, we are interested in cardiac cell models because the suppression of arrhythmia through studies of mathematical models can reduce the risk of sudden cardiac arrest. A recent model consists of 45 ordinary differential equations [2]. Usually, such mathematical models are described by nonlinear equations. The nonlinear differential equations cannot be solved analytically. Thus, numerical analysis using computers is effective. In numerical analysis, more accuracy needs more computational costs. Moreover, detailed mathematical models also require much time for simulation. Thus, we must construct a fast numerical calculator with low electric power consumption.

In nonlinear dynamical systems, we observe complicated phenomena such as quasi-periodic states, chaotic states and co-existence of attractors. The generation of these phenomena is related to bifurcations. The bifurcation is defined by the qualitative change of the solutions due to a perturbation of parameter values. It is very important to understand bifurcation structure in parameter space because dynamical diseases such as Cheyne-Stokes respiration and chronic granulocytic leukemia are caused by bifurcations [3]. However, studying parameter space in complicated dynamical system (high-dimensional dynamical system) needs much time because we should investigate bifurcations in many combinations of parameters contained in the system.

In this paper, we design nonlinear analysis tools for bifurcations and basins of attraction using FPGA (Field Programmable Gate Array). FPGA is a reconfigurable integrated circuit, and it is good at parallel processing [4, 5, 6]. By numerical experiment, we show that our method is faster than the conventional numerical calculation using CPU.

2. Preliminaries

2.1. Fixed-point Operation

In this research, we use the 32 bits (1, 7 and 24 bits for the sign, integer and fractional part) fixed-point number representation for operation. Our FPGA has a multiplier with 18 bits. We use the Booth algorithm and the multiplier on FPGA for multiplication. The multiplier on FPGA is faster than the Booth method; however, accuracy becomes worse. Thus, we compare these methods on FPGA to check the relationship between speed and accuracy. The CORDIC algorithm [7] is used for the calculation of mathematical functions.

2.2. Model Equations

Duffing's equations which describe the dynamics of the oscillatory circuit with a nonlinear inductor [8] are given by:

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -ky - c_3x^3 - B_0 + B\cos\omega t.\end{aligned} \tag{1}$$

We fix the parameter values as $k = 0.1$, $c_3 = 1.0$, $\omega = 1$, and $B_0 = 0.075$.

The BVP (Bonhöffer-van der Pol) model [9] with an external force is given by:

$$\begin{aligned}\frac{dx}{dt} &= c(y + x - \frac{1}{3}x^3 + h\sin\omega t) \\ \frac{dy}{dt} &= -\frac{1}{c}(x + by - a).\end{aligned} \tag{2}$$

We fix the parameter values as $a = 0.0$, $b = 0.4$, and $c = 1.5$. This model was proposed as simplification of the Hodgkin-Huxley neuron model and its bifurcation structures of the single model [10, 11] and the coupled models [12, 13, 14] were widely studied.

We define the Poincaré map for Eqs. (1) and (2) as

$$T : R^2 \to R^2;\ \xi \mapsto T(\xi, \lambda) = \psi(2\pi/\omega, \xi, \lambda),$$

where $\xi = [x, y]^{\mathrm{T}}$, λ is a parameter vector, $\psi(t, \xi_0, \lambda)$ is assumed to be a solution of Eq. (1) or (2) with an initial condition $\xi_0 = [x_0, y_0]^{\mathrm{T}}$ at $t = t_0$. Then the l-periodic point ξ^* of Eq. (1) or (2) is defined as

$$\xi^* - T^l(\xi^*, \lambda) = 0. \quad (3)$$

3. Results

We use the computer (CPU: i5-560m with clock frequency 2.66 GHz) for software calculation (compiler: Borland bcc55) and the FPGA board (DE2-115). This board has the VGA port, so results (diagrams) are shown on a display through this port. FPGA (Cyclone IV E: clock frequency is 50 MHz and logic elements are 114,480) is equipped on the board.

3.1. Basin of attraction

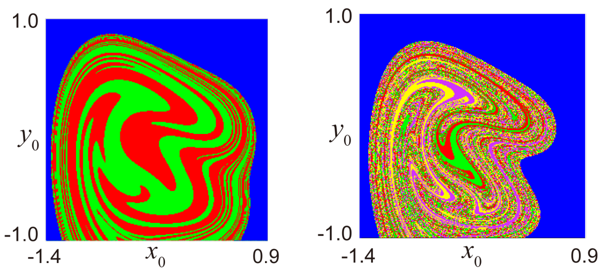
If attractors of the system coexists, then the basin of attraction gives us useful information of initial state dependency. A plane of initial states on a grid is considered, and we check which attractor is obtained from each initial point. We put distinct color for attractors. Usually, the basin boundary gives a stable manifold of a saddle type periodic solution. Thus, using the basin we can visualize the global structure of stable manifolds. Moreover, its fractalization is related to homoclinic points [15, 16], and its qualitative change is related to synchronization [17].

Figures 1, 2, and 3 show the basins of attraction for Duffing's equations using the software, FPGA (Booth method) and FPGA (multiplier), respectively. There coexist a fixed point and two-periodic points at $B = 0.15$. From the initial states colored by blue, we observe the fixed point as an attractor. Two-periodic points are obtained from the initial states colored by red and green. Increasing the value of B to 0.185, four-periodic points appear as a result of the period-doubling bifurcation of the two-periodic points. The boundaries of four colors are fractalized because of the appearance of homoclinic points [18]. Comparing between the results of the software and FPGA, the shape of each colored region is almost the same; FPGA can calculate the basin of attraction. Table 1 shows the comparison of runtime for the software, FPGA (Booth), and FPGA (multiplier). When the degree of parallelism is one, the software is faster than

Table 1: Results for Duffing's equations

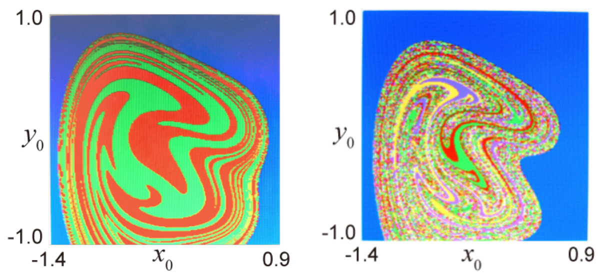
	degree of parallelism	runtime (hh:mm:ss)	occupied LE (%)
software	1	01:49:7	-
FPGA (Booth)	1	22:55:44	4
	2	11:27:52	6
	44	00:31:16	99.8
FPGA (multiplier)	1	16:13:05	3
	2	08:06:32	5
	54	00:18:02	99.8

FPGA. When the degree is two, FPGA (Booth) and FPGA (multiplier) occupied 6 and 5% of total logic elements (LE). Considering the increasing rate from the no parallel processing, FPGA (Booth) and FPGA (multiplier) can have 44 and 54 parallel processings at most. Thus, runtimes are estimated to be 31 and 18 minutes, which is 3.5 and 6.1 times faster than using software.



(a) $B = 0.15$. (b) $B = 0.185$.

Figure 1: Basin of attraction of Duffing's equations obtained by using software.

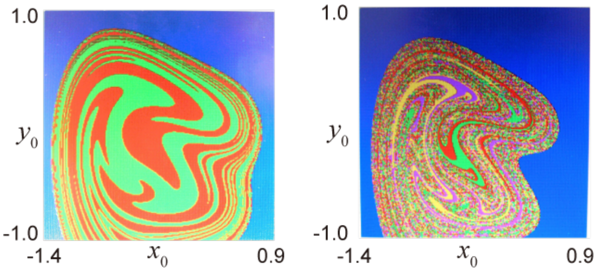


(a) $B = 0.15$. (b) $B = 0.185$.

Figure 2: Basin of attraction of Duffing's equations obtained by using FPGA (Booth).

3.2. Bifurcation diagram

We show results of calculating bifurcation diagrams for the BVP model. The meaning of colors is shown



(a) $B = 0.15$. (b) $B = 0.185$.

Figure 3: Basin of attraction of Duffing's equations obtained by using FPGA (multiplier).

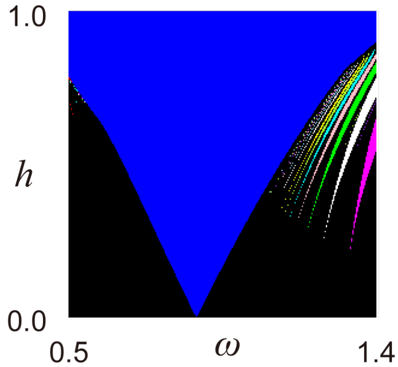
Table 2: Color assignment

period	color	period	color
1	blue	7	white
2	red	8	slate blue
3	magenta	9	pink
4	green	10	purple
5	cyan	non-period or divergence	black
6	yellow		

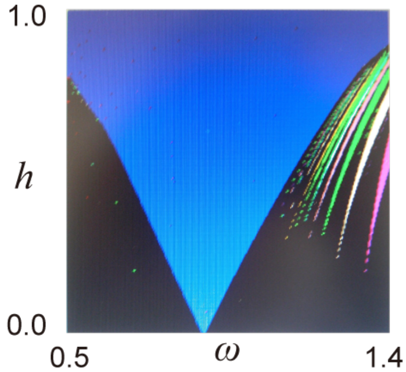
in Tab. 2 [19]. Figures 4(a), 4(b), and 4(c) represent the result of using the software, FPGA (Booth), and FPGA (multiplier), respectively. Comparing these figures, we can see that both methods of FPGA obtain the almost same result as the software's. The BVP model without the external force has an oscillatory solution at the parameter values we decided. The parameter region of the entrainment of the frequency for the oscillatory solution is colored by blue. Other colors indicate the regions of the existence of stable sub-harmonic oscillations. The results of runtime are shown in Tab. 3. The degree of parallelism is changed from 1 to 2, occupation of the FPGA (Booth) and FPGA (multiplier) are 3 % and 2 % raise. Thus, theoretically 32 and 61 degrees of parallelism are possible, and it is 3.2 and 22 times faster than using the software.

Table 3: Results for BVP model

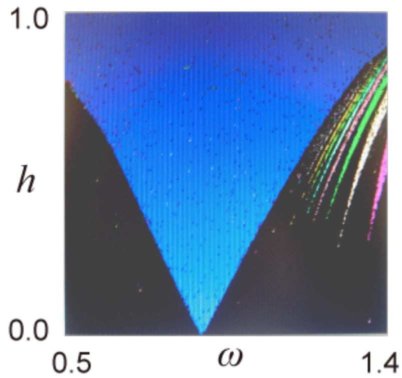
	degree of parallelism	runtime (hh:mm:ss)	occupied LE (%)
software	1	02:17:14	-
FPGA (Booth)	1	22:55:44	5
	2	11:27:52	8
	32	00:42:59	99.8
FPGA (multiplier)	1	06:20:17	3
	2	03:10:09	5
	61	00:06:14	99.8



(a) Software.



(b) FPGA (Booth)



(c) FPGA (multiplier)

Figure 4: Bifurcation diagram of BVP model.

4. Conclusion

We developed nonlinear analysis tools on FPGA. The calculators of bifurcation diagrams and basins of attraction were constructed on FPGA. We confirmed the accuracy of our calculators and showed that our method is faster than the conventional method using CPU. Also, the calculator on FPGA has an advantage of using less electric power. We are now trying to use OpenCL to combine different platforms such as CPU, GPU, DSP, and FPGA.

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