Adaptive Stabilization of a Saddle Steady State of a Conservative Dynamical System: a Spacecraft at the Lagrange Point L2 of the Sun–Earth System

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Abstract—An adaptive feedback method for tracking and stabilizing unknown or slowly varying saddle type steady states of conservative dynamical systems is described. We demonstrate that a conservative saddle point can be stabilized with neither unstable nor stable filter technique. Meanwhile, a controller, involving both filters in parallel, works perfectly. As a specific example, the Lagrange point L2 of the Sun–Earth system is discussed and a general second order saddle model is considered.

1. Introduction

In astrodynamics the stability of the spacecrafts at the Lagrange points L1 and L2, also in related to them the Lyapunov and the Lissajous orbits is a very challenging problem. For example, several recent space missions use the Lissajous orbit around the L2 of the Sun–Earth system. Specifically, the Lagrange points belong to the wide class of the saddle type unstable steady states (USS) of conservative dynamical systems. A straightforward idea to stabilize an USS is to apply a proportional feedback force. Classical control methods, using proportional feedback, require a mathematical model of a system or at least the location of the USS in the phase space for the reference point. However, in many practical situations neither the exact models nor the coordinates of the reference point are accessible. Moreover, the position of a reference point may slowly vary with time in an unpredictable manner because of external perturbations. Therefore adaptive, that is model-independent and reference-free methods, automatically locating the USS are preferable.

The simplest adaptive technique for stabilizing the USS is based on the derivative controller [1, 2, 3]. A feedback force in the form of a derivative kdx/dt derived from an observable $x(t)$ introduces artificial dynamical losses and consequently damps the oscillations. An important feature of the derivative is that it does not influence the position of the USS, since it vanishes when the variable $x(t)$ approaches the goal state. The method has been successfully applied to diverse nonlinear dynamical systems, for example to stabilize a laser [1], a chaotic Chua circuit [2] and an electrochemical reaction [3]. However, the technique is rather sensitive to high-frequency noise unavoidably present in the experimental signal $x(t)$, since it requires a differentiation of the observable.

More advanced adaptive method for stabilizing the USS employs low- (high-) pass filter in the feedback loop [4, 5, 6, 7, 8, 9]. Provided the cut-off frequency of the filter is low enough, the filtered image $v(t)$ of the observable $x(t)$ asymptotically approaches the USS and therefore can be used as a reference point in the proportional feedback. This method has been verified in several experimental systems, including electrical circuits [4, 5, 6, 7] and lasers [8, 9].

Two more techniques, though originally designed to control unstable periodic orbits, can be used to stabilize the USS as well. The first is the time-delayed feedback method proposed by Pyragas [10, 11]. Under appropriate choice of the delay value the method is able to stabilize the steady states [5, 6, 12]. The second one is the notch-filter method [13]. Though developed to stabilize periodic orbits, it is also capable to control the steady states in the case at least two notch filters with different and incommensurate resonance frequencies are applied.

However, all the mentioned techniques, as well as the recently suggested modification with the Taylor predictor [14] and extention to strongly nonlinear regions [15], are restricted to unstable nodes and unstable spirals only. They fail to stabilize the saddle type states (the USS with an odd number of real positive eigenvalues).

To get around the problem of the odd number limitation Pyragas *et al.* [16, 17] proposed to use an unstable filter, that is a bold idea to fight one instability with another instability. The technique has been demonstrated to stabilize saddles in several mathematical models [16, 17, 18, 19] also in the experiments with

an electrochemical oscillator [16, 17] and the Duffing–Holmes type electronic circuit [18, 19].

Unfortunately, this advanced method is limited to *dissipative* dynamical systems only. It is not applicable to *conservative* systems. The situation is somewhat similar to the problem of the famous OGY method [20, 21], in the sense that it does not work in the Hamiltonian systems. The limitation of the Pyragas' unstable filter method can be proved analytically using the well-known Hurwitz stability criteria. According to these criteria the necessary condition for stabilizing a saddle state is that the cut-off frequency of the applied unstable filter is lower than the damping coefficient of the system under control. While damping is zero (!) in the conservative systems under definition. Formally, to fulfill this stability criteria, the cut-off frequency could be set negative. However, this would mean that the unstable filter should become a stable one and, therefore, inappropriate to stabilize a saddle.

In this paper, we demonstrate that the conservative saddles can be successfully stabilized by means of the recently proposed conjoint filter [22], that involves both an unstable and stable subfilters. Previously such a combined filter has been employed to overcome the problem of latencies in the feedback loop when stabilizing saddles in the dissipative systems.

2. Lagrange point L2

We consider dynamics of a body of mass μ, e.g. a spacecraft at the Lagrange point L2 of the Sun–Earth system (Fig. 1). The dynamics taking into account the centrifugal force and the forces of gravity is given by

$$\mu\ddot{R} = \mu\Omega^2(R_0 + R) - \frac{\gamma\mu M}{(R_0 + R)^2} - \frac{\gamma\mu m}{R^2} + P. \quad (1)$$

Here γ is the gravitational constant, M and m is the mass of the Sun and the Earth, respectively, R_0 is the distance of the Earth from the Sun, R is the distance of the Lagrange point L2 from the Earth, P is in general unknown external force, which is either constant or slowly varying in time. Since the Lagrange point L2 lies on the same line as the Sun and the Earth, the angular velocity Ω in the centrifugal force of the body
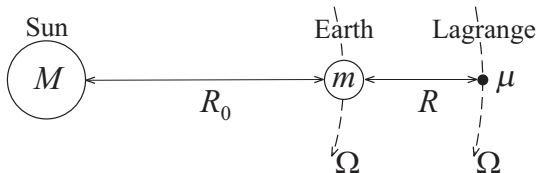
at the Lagrange point is just the same as that of the Earth: $\Omega^2 = \gamma M/R_0^3$. After introducing the dimensionless quantities $r = R/R_0$ and $\varepsilon = m/M$ Eq. (1) can be presented as

$$\ddot{r} - \Omega^2 F(r, \xi) = 0, \quad (2)$$

$$F(r, \xi) = 1 + r - \frac{1}{(1 + r)^2} - \frac{\varepsilon}{r^2} + \xi, \quad (3)$$

$$\xi = \frac{P}{\mu R_0 \Omega^2}. \quad (4)$$

The nonlinear function $F(r, \xi)$ is depicted in Fig. 2. The steady state of the system r_0 can be found from an algebraic equation $F(r_0, \xi) = 0$. The value of r_0 can be roughly estimated from a simple formula $r_0 \approx (\varepsilon/3)^{1/3}$, which is valid for $(r_0, \xi) \ll 1$. Linearization of the system around the steady state point $(r = r_0 + x, |x| \ll r_0)$ yields:

$$\ddot{x} - \Lambda^2 x = 0, \quad \Lambda^2 = \Omega^2 F'(r_0, \xi). \quad (5)$$

Here $F'(r_0, \xi)$ is the derivative of $F(r, \xi)$ with respect to r at the point r_0. We note, that $F'(r_0, \xi) > 0$. Consequently, Λ^2 is always positive and the eigenvalues of the corresponding characteristic equation $\lambda^2 - \Lambda^2 = 0$ are $\lambda_{1,2} = \pm\Lambda$, confirming that the steady state r_0 is the saddle type USS. The value $\lambda_1^{-1} = \Lambda^{-1}$ can serve as an estimate of the characteristic runaway time τ_L from the Lagrange point: $\tau_L = \Lambda^{-1} = \Omega^{-1}/\sqrt{F'(r_0)} \approx Y_E/6\pi$, where Y_E is the orbital period of the Earth around the Sun, that is the Earth year; thus the $\tau_L \approx 19$ days. Further, time in Eq. (5) can be normalized to the value of Λ^{-1}, so that the equation becomes model-independent and describes behavior of any conservative dynamical system in the vicinity of a saddle point:
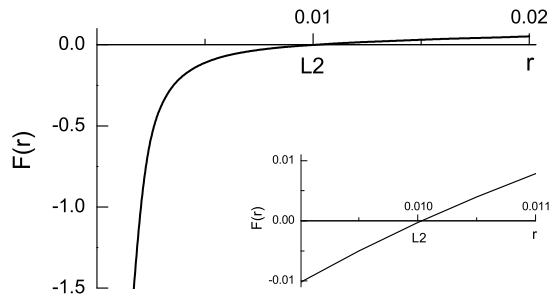
$$\ddot{x} - x = 0. \quad (6)$$

Figure 2: Nonlinear function $F(r)$ with $\varepsilon = 3.10^{-6}$. The insert shows that the force $F(r)$ is nearly linear in the range from 0.009 to 0.011, that is around the Lagrange point $r_0 = 0.01 \pm 0.001 = 0.01 \pm 10\%$.

Figure 1: Lagrange point L2 of the Sun–Earth system. The Sun diameter, the Earth diameter, the distances R_0 and R are not in scale.

3. Stabilizing a body at the Lagrange point

First of all we apply the unstable filter [16, 17] trying to stabilize the saddle point:

$$\ddot{x} - x = k_1(u - x), \qquad (7)$$
$$\dot{u} = \omega_1(u - x). \qquad (8)$$

The corresponding characteristic equation is

$$\lambda^3 - \omega_1\lambda^2 + (k_1 - 1)\lambda + \omega_1 = 0. \qquad (9)$$

There is a considerable drop of the largest $\text{Re}\lambda$ with k_1, however it remains positive indicating instability of the closed loop (Fig. 3a).

Though the stable filter technique is not expected to stabilize a saddle steady state, we consider it here for comparison:

$$\ddot{x} - x = k_2(v - x), \qquad (10)$$
$$\dot{v} = \omega_2(x - v). \qquad (11)$$

From its characteristic equation

$$\lambda^3 + \omega_2\lambda^2 + (k_2 - 1)\lambda - \omega_2 = 0. \qquad (12)$$

one can make sure that the result is practically the same (Fig. 3b) as for the unstable filter. The controller fails to stabilize the saddle.

However, when combined in parallel:

$$\ddot{x} - x = k_1(u - x) + k_2(v - x), \qquad (13)$$
$$\dot{u} = \omega_1(u - x), \qquad (14)$$
$$\dot{v} = \omega_2(x - v), \qquad (15)$$

the two filters give unexpectedly excellent result as evident from the solution of the characteristic equation

$$
\begin{aligned}
\lambda^4 \;+\; & (\omega_2 - \omega_1)\lambda^3 - (k_1 + k_2 - 1 - \omega_1\omega_2)\lambda^2 \\
+\; & [k_1\omega_2 - k_2\omega_1 - (\omega_2 - \omega_1)]\lambda \qquad (16) \\
+\; & \omega_1\omega_2 = 0.
\end{aligned}
$$

Indeed, the largest eigenvalue crosses zero at a certain value of the the feedback coefficient k_1 (Fig. 3c). The stability properties can be also checked using the Hurwitz criteria.

4. Concluding remarks

We have suggested using an adaptive control method for stabilizing unknown and slowly varying saddle type steady states of conservative dynamical systems. The controller is model-independent and reference-free. It does not require knowledge of either the mathematical model or the position of the steady state, but automatically tracks the state and stabilizes it. The controller involves both, the unstable filter and the stable filter in the feedback loop. While seperately
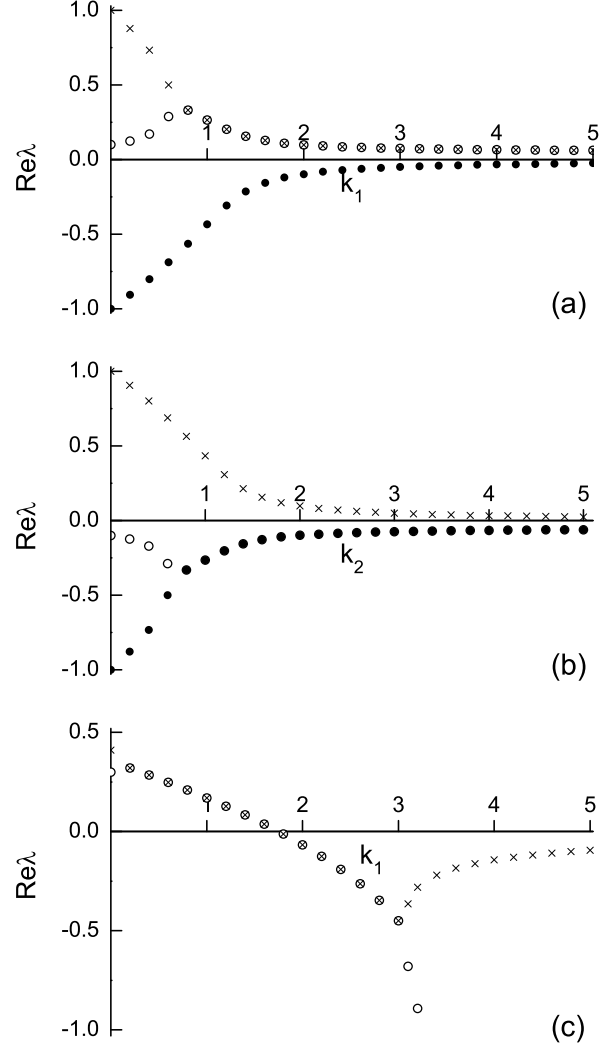

Figure 3: Dependence of the real part of the eigenvalues $\text{Re}\lambda$ on the control gains k_i. (a) unstable filter only with $\omega_1 = 0.1$. (b) stable filter only with $\omega_2 = 0.1$. (c) unstable and stable filters in parallel with $\omega_1 = 0.3$, $k_2 = 15$, $\omega_2 = 7$.

each of the filters seems to be useless and senseless in the case of conservative dynamical systems, when combined in one they give an excellent stabilizing result. In the nearest future we going to construct an undamped electronic circuit, imitating the dynamical behaviour of a body at the Lagrange point of the conservative Sun–Earth system. The experimental results will be published elsewhere.

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