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Analysis of Digital Spike Maps Based on Simple Feature Quantities

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Abstract—The digital spike map is a simple digital dynamical system on a set of finite number of lattice points. Depending on parameters and initial condition, the map can exhibit various periodic/transient spike-trains. In order to analyze the map, this paper presents two simple feature quantities. The first one can characterize plentifulness of the periodic spike-trains. The second one can characterize deviation of the transient phenomena. Using the two quantities, we demonstrate several fundamental results.

1. Introduction

The digital spike map (DSM) is a simple dynamical system on a set of a finite number of lattice points. Depending on parameters and initial condition, the DSM can generate a variety of spike-trains ([1] and Refs. therein). The DSM can be ragarded as a digital version of the analog 1D maps [2]. The DSM can have a variety of periodic spike-trains (PSTs) as steady states. The PSTs can co-exist and the DSM exhibits either PST depending on the initial value.

As motivations for studying the DSM include the following two points. First, the DSM can be a simple and typical example digital dynamical systems such as cellular automata, dynamic binary neural networks, and digital spiking neurons [3]-[6]. Such systems have been studied not only for analysis of its nonlinear dynamics but also for variety of applications such as image processing and information compression [7] [8]. Second, the DSM can be a basic system for spike-based engineering applications such as spike-based communication and information processing [9]-[11].

This paper presents two simple feature quantities for analysis of the DSMs and investigate several examples of the DSMs. The first quantity is the rate co-existing PSTs. The DSM can have plural PSTs and exhibits either of them depending on the initial condition. It can characterize plentifulness of the PSTs. The second quantity is the concentricity of transition to the PSTs. This quantity can characterize deviation of the transient phenomena to the PSTs and is based on the concentricity of state transition [12]. Using the two quantities, we then demonstrate fundamental results for DSMs based on the bifurcating neuron (BN). The BN is a simple analog dynamical system that can generate various chaotic/periodic spike-trains [13]-[15].

2. Digital Spike Map

Fig. 1 shows the spike-position map on a set of lattice points

$$\begin{aligned} &\tau_{n+1} = F(\tau_n),\ F : L \to L \\ &\tau_n \in L \equiv \{l_1, l_2, \cdots\},\ l_i \equiv (i-1)/N,\ i = 1, 2, \cdots \end{aligned} \tag{1}$$

where we assume that the system has normalized period 1 and devide L into subintervals of each period:

$$\begin{aligned} &L = \textstyle\bigcup_{n=0}^{\infty},\ L_1 \equiv \{l_1, l_2, \cdots l_N\} \\ &L_k \equiv L_1 + k - 1 \end{aligned}$$

We also assume that the map satisfies periodic condition and the law of cause and effect.

$$F(\tau + 1) = F(\tau) + 1,\ F(\tau) > \tau \tag{2}$$

Let τ_n denote the n-th spike-position on L As an initial spike train τ_1 is given as shown in Fig. 1, the map generate a spike-train characterized by the spike-positions

$$Y(\tau) = \begin{cases} 1 & \text{for } \tau = \tau_n \\ 0 & \text{for } \tau \neq \tau_n \end{cases} \tag{3}$$

Note that the spike-positions are restricted on the lattice points. This map can generate a variety of spike-train. For simplicity, we assume that the initial spike-position is given in the first subinterval and that initial

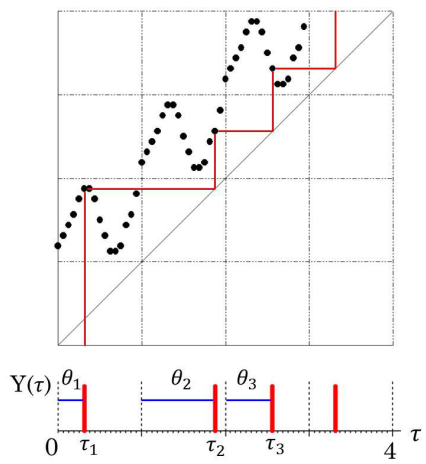

Figure 1: Digital spike-position map and spike-train

condition generates a spike-train such that one spike exists in each subinterval

$$\tau_1 \in L_1,\ F(L_k) \in L_{k+1},\ k = 1, 2, \cdots$$

Introducing the spike phase, $\theta_n = \tau_n \bmod 1$, we define the digital spike-phase map from L_1 to itself

$$\theta_{n+1} = f(\theta_n) \equiv F(\tau_n) \bmod 1,\ f : L_1 \to L_1 \qquad (4)$$

Hereafter, we refer to this map as the digital spike map (DSM). The iteration of Eq. (4) generates a sequence of the phases $\{\theta_n\}$ and the sequence defines a sequence of spike-positions: a spike-train.

$$\tau_n = \theta_n + n - 1 \text{ for } \theta_1 = \tau_1 \in L_1$$

Since the number of lattice points is finite, the steady state must be periodic. Note that the initial spike-position (spike-phase) in given in L_1 and one initial spike-position gives one spike-train.

Definition 1 (see Fig. 2): A point $p \in L_1$ is said to be a periodic point (PEP) with period k if $p = f^k(p)$ and $f(p)$ to $f^k(p)$ are all different where f^k is the k-fold composition of f. The PEP with period 1 is referred to as a fixed point. A sequence of the PEPs $\{p, f(p), \cdots, f^{k-1}(p)\}$ is said to be a periodic orbit (PEO) with period k. A PEP corresponds to an initial spike-position in L_1 that gives one PST. Hence one PEP with period k corresponds to one PST with period k and one PEO with period k corresponds to k PSTs with period k.

3. Feature Quantities

Since the domain L_1 consists of N lattice points and each lattice point has one image, the DSM has N^N variations. Depending on parameters and initial condition, the DSN can exhibit extremely complicated dynamics and it is very hard to give general theory for the dynamics. This paper tries to consider dynamics of typical examples of DSMs. In order to classify and consider the dynamics, we introduce two simple feature quantities.

Let N_p be the number of PEPs on N lattice points. The first feature quantity is the rate of PEPs (PSTs):

$$\alpha = \frac{\#\text{PEP}}{N} = \frac{N_p}{N},\ 1/N \le \alpha \le 1 \qquad (5)$$

This quantity characterize plentifulness of steady states (PSTs).

In order to define the second feature quantity, we define the initial point histogram (IPH).

Definition 2: Let a DSM has N_p pieces of PEPs. Let p_i be the i-th PEP, $i = 1 \sim N_p$. We classify PSTs for its initial spike-position $\tau_1 \in [0, 1)$. If a PST started from $\tau_1 = p_i \in L_1$ then the PST is referred to as the i-th PST. The transient states for the i-th PST is characterized by EPP(s) falling into the i-th PST. The IPH is frequency of initial points falling into the each PST:

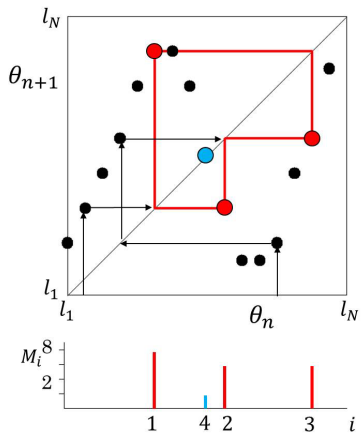

Figure 2: Disital sipke map and initial points histogram.

$$M_i = \#\text{initial points falling into the } i\text{-th PST} \qquad (6)$$

The second feature quantity is the concentricity of transition to the PSTs:

$$\beta = \sum_{i=1}^{N_P} \left(\frac{M_i}{N}\right)^2,\ 1/N \le \beta \le 1 \qquad (7)$$

In Fig. 2, the IPH of $N = 16$ is constructed for 4 pieces of PEPs. The 1st to 3rd PEPs ($i = 1 \sim 3$) construct a PEO with period 3 and the 4th PEP ($i = 4$) is the fixed point. Since $M_1 = 7$, $M_2 = 4$,$M_3 = 4$,and $M_4 = 1$, we obtain $\beta = 82/256$. The IPH characterizes the basin of attraction to the PSTs and The quantity β characterizes the variation of the basin.

Using the two feature quantities α and β, we construct steady versus transient plot (ST-plot) as shown in Fig. 3. In the figure, two examples of the DSM and IPH for $N = 16$ and $N_p = 4$ are shown. In Fig. 3 (a), the DSM has three PEPs with period 3 and one fixed point. The IPH is uniform and we obtain $\alpha = 1/4$ and $\beta = 1/4$. In Fig. 3 (b), the DSM has four fixed points and all the EPPs fall into the third fixed point ($i = 3$). The IPH is delta-function-like and we obtain $\alpha = 1/4$ and $\beta = 172/256$.

For the ST-plot, we introdue two characteristic curves. If the IPH is uniform the feature quantities are plotted on

$$\text{The uniform curve: } \alpha\beta = \frac{1}{N} \qquad (8)$$

If all the initial poins (not PEPs) fall into one PEP, then the feature quantities are plotted on

$$\text{The concentrate curve: } \beta = (1-\alpha)^2 + \frac{2-\alpha}{N} \qquad (9)$$

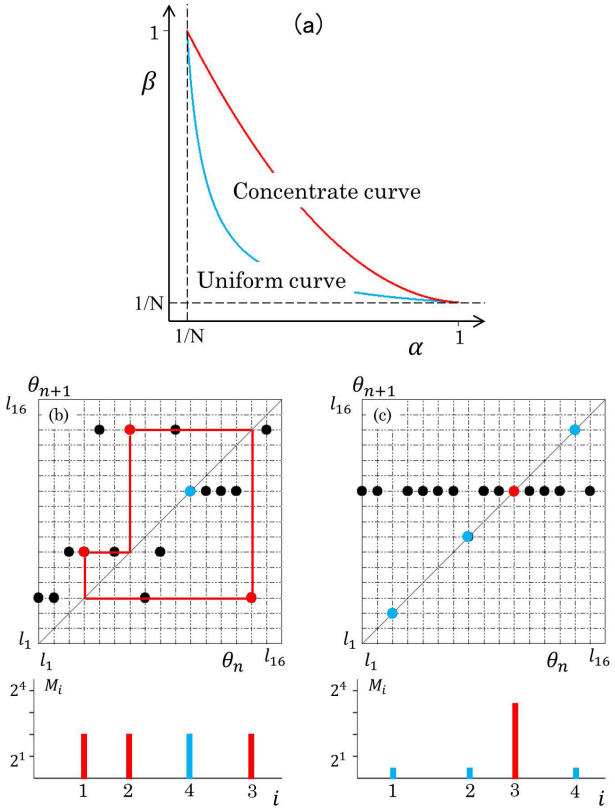

Figure 3: (a) ST plot. (b) DSM/IPH on the uniform curve. (c) DSM/IPH on the concentrate curve. ($N = 16$, $N_p = 4$).

The endpoints of the two curves are $(1/N, 1)$ where the DSM has only one fixed point and $(1, 1/N)$ where the all the lattice points are PEPs. The feature quantities must not be plotted out of the region surrounded by the uniform curve and concentrate curve,

4. Numerical experiments

Here we analyze examples of DSM based on the analog spike map (ASM):

$$\theta_{n+1} = g_a(\theta_n) = \begin{cases} a\theta_n & \text{for } 0 \le \theta_n < d_1 \\ -a_2(\theta_n - \frac{1}{2}) + \frac{1}{2} & \text{for } d_1 \le \theta_n < d_2 \\ a(\theta_n - 1) + 1 & \text{for } d_2 \le \theta_n < 1 \end{cases}$$

where $0 < d_1 < 1/2$, $d_2 \equiv 1 - d_1$, and $a_2 = (2ad_1 - 1)/(1 - 2d_1)$. This ASM is deribed from the bifurcating neuron (BN). Discretizing the ASM, we obtain the DSM:

$$\theta_{n+1} = \frac{1}{N}\mathrm{INT}(Ng_a(\theta_n) + 0.5) \equiv g_d(\theta_n) \qquad (10)$$

where $\theta_n \in L_1$ and $\mathrm{INT}(X)$ means the integer part of X. For simplicity, this paper studies the case

$$d_1 = 1/3,\ (a_2 = 2a - 3),\ 3/2 < a < 3,\ N = 32 \quad (11)$$

In this case, the ASM is expanding and exhibits chaotic spike-trains.

Case 1: α is very small and β is very large. Fig. 4(a) shows a typical example of DSM for $a = 2.35$. This DSM has two fixed points l_1 and l_{16}. All the other points are EPPs. The fixed point corresponds to a PST with period 1. This DSM has #PST = 2, $\alpha = 0.07$. All the EPPs fall into one fixed point l_{16} and The concentricity of transition to the PEPs is very large: $M_1 = 1$, $M_2 = 31$ and $\beta = \frac{1^2+31^2}{32^2} \simeq 0.94$, On the ST-plot of Fig. 7 (α, β) closed to the endpoints $(1/N.1)$.

Case 2: α is very large and β is very small. Fig. 4(b) shows a typical example of DSM for $a = 3.0$. All the points are PEPs. This DSM has #PST = 32, $\alpha = 1.0$. This DSM has 6 PEOs. two fixed points, PEO with period 2 (2PSTs), PEO with period 4 (4PSTs), PEO with period 8 (8PSTs), PEO with period 16 (16PSTs). The DSM has no EPP. (Fig. 5) $M_1 = M_2 = \cdots = M_{32} = 1$ and $\beta = 1$. As shown in Fig. 7, (α, β) is plotted at the end point $(1, 1/N)$ of two characteristic curves on the ST-plot.

Case 3: α and β are small. Fig. 6(a) shows a typical example of DSM for $a = 2.24$. This DSM has 2 fixed points and 4 PEPs with period 2. The DSM has 6 PEPs and $\alpha = 0.19$. As shown in Fig. 7, (α, β) is plotted between the uniform and concentrate curves.

Case 4: α is large and β is small. Fig. 6(b) shows a typical example of DSM for $a = 2.38$. This DSM has 2 fixed points and 10 PEPs with period 5. The DSM has 14 PEPs and $\alpha = 0.19$. As shown in Fig. 7, (α, β) plotted almost on the uniform curve.

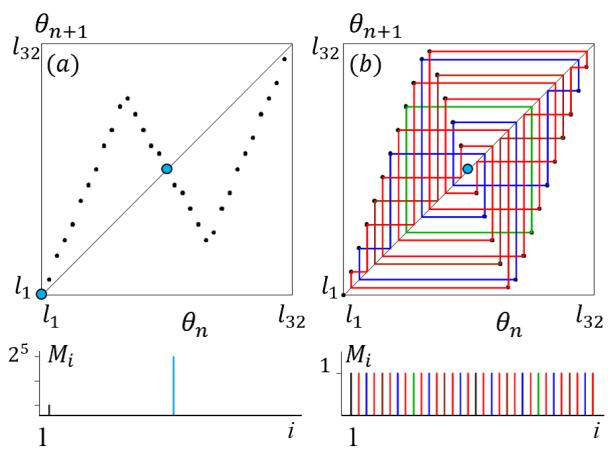

Figure 4: (a)DSM and IPH for $a = 2.35$ and $N = 32$. $\alpha = 0.07$, $\beta = 0.94$. (b)DSM and IPH for $a = 3.0$ and $N = 32$. $\alpha = 1.0$, $\beta = 1/32$.

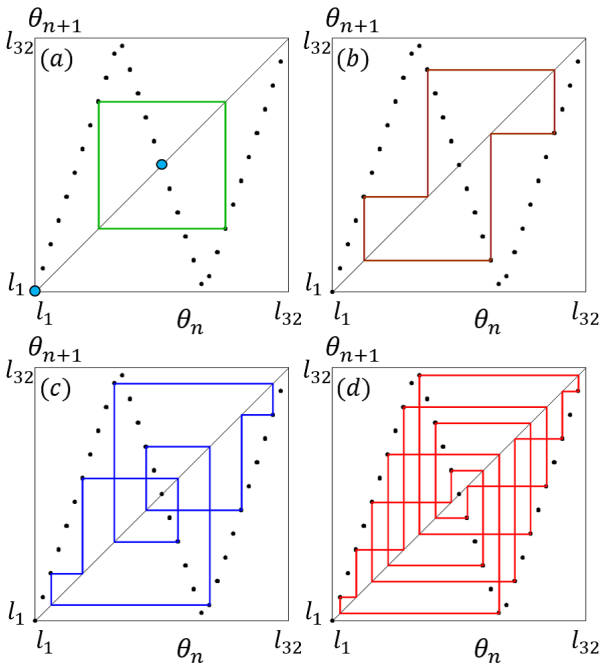

Figure 5: DSM for $a = 3.0$ and $N = 32$. (a) fixed points and PEO with period 2. (b) PEO with period 4. (c) PEO with period 8. (d) PEO with period 16.

5. Conclusions

In order to analyze the DSM, two feature quantities, α and β, are presented in this paper. The ST-plot is useful for classification and investigation of the dynamics of DSMs. Using typical examples based on the BN, basic results are demonstrated. Future problems include analysis of bifurcation phenomena of the DSMs and classification of the PSTs on the ST-plot.

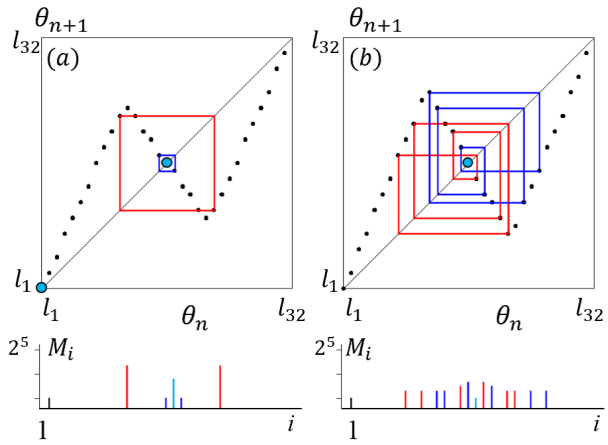

Figure 6: (a)DSM and IPH for $a = 2.24$ and $N = 32$. $\alpha = 0.19$, $\beta = 0.31$. (b)DSM and IPH for $a = 2.38$ and $N = 32$. $\alpha = 0.44$, $\beta = 0.08$

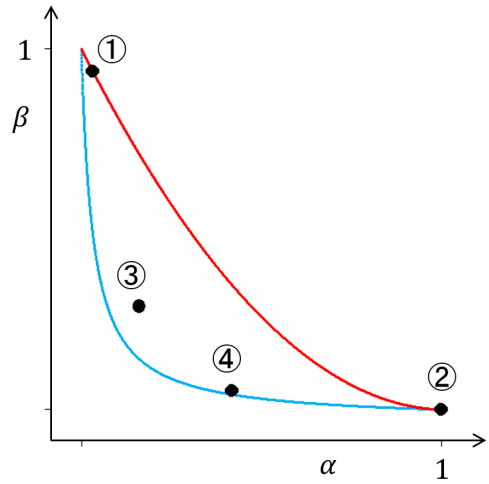

Figure 7: ST-plot for $N = 32$

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