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On Quasi Consensus of Heterogeneous Second-order Multi-agent Systems With Nonlinear Dynamics

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Abstract—In this paper, quasi consensus of second-order leader-following heterogeneous multi-agent systems has been investigated. A sufficient criterion for guaranteeing quasi consensus has been derived. Furthermore, some simulations are given to illustrate the effectiveness of the theoretical theorems.

1. Introduction

Heterogeneous complex systems are ubiquitous in real world systems such as society, technology, biology and nature. Therefore, the collective behaviours of heterogeneous complex systems have attracted an increasing interest [1, 2, 3, 4, 5]. The heterogeneous complex systems are composed of different linked nodes. [1] studied the robustness of output synchronization in heterogeneous agent dynamics. [3] studied quasi synchronization of a heterogeneous system, i.e., the states of all nonidentical nodes can not reach synchronization but reach an bounded error with the weighted average of all node states as the synchronization target. By introducing a goal state and an impulsive algorithm, [4] studied complete synchronization of a heterogeneous coupled Duffing oscillation systems. Similar to synchronization [6], consensus in multi-agent system is the other typical collective behaviour of complex systems [7, 8, 9, 10, 11, 12, 13]. To achieve the consensus of multi-agent systems, some controllers are usually added. Since complex system has a number of nodes, it is very difficult to add controller to all nodes of complex systems. Pinning control is a reasonable choice [10, 11].

Few papers are focused on consensus of heterogeneous multi-agent systems with nonlinear dynamics. Motivated by the above work and discussions, this paper studies quasi consensus of a second-order leader-following heterogeneous multi-agent system. By adopting a pinning control, a sufficient criterion for guaranteeing quasi consensus has been derived. It has been proved that all the followers will keep a bounded error with the leader.

Throughout this paper, we adopt the following notations. Let R^n denote the n-dimensional Euclidian space. Let O_N and I_N be the $N \times N$ zero matrix and identity matrix, respectively. If M is symmetric, then $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ be its minimum and maximum eigenvalues of M, respectively. For a symmetric M, $M > 0$ ($M < 0$) means that M is positive definite (negative definite). Let 1_N and 0_N denote the $N \times 1$ column vectors of all ones and all zeros respectively. The symbol $\otimes$ denotes the Kronecker product.

2. Preliminaries

An undirected graph [14] denotes $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ consists of a vertex set $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and a set of undirected edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix $\mathcal{A} = [a_{ij}]$ with nonnegative entries. An undirected edge of graph $\mathcal{G}$ is an unordered pair of distinct vertices. We use $e_{ij} = (v_i, v_j)$ to denote an undirect edge, meaning that nodes v_i and v_j can exchange information with each other. The weighted adjacency matrix $\mathcal{A} = [a_{ij}]$ are defined as follows: $a_{ij} = a_{ji} > 0$ if there is an edge between nodes v_i and v_j; otherwise $a_{ij} = 0$. We assume that $a_{ii} = 0$. The Laplacian matrix $\mathcal{L} = [l_{ij}]$ associated with adjacency matrix $\mathcal{A}$ is defined as follows: $l_{ij} = -a_{ij}$ if $i \neq j$ and $l_{ii} = \sum_{j=1, j\neq i}^{N} a_{ij}$, which ensures the property that $\sum_{j=1}^{N} l_{ij} = 0$. For an undirected graph, $\mathcal{A}$ and $\mathcal{L}$ are symmetric.

Consider a second-order heterogeneous multi-agent systems with nonlinear dynamics [7, 8, 9, 10, 11, 12]

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = f_i(t, x_i(t), v_i(t)) + u_i(t), \end{cases} \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^\top$ and $v_i(t) = (v_{i1}(t), v_{i2}(t), \cdots, v_{in}(t))^\top$ are position and velocity states of agent i, respectively. $f_i(t, x_i, v_i) = (f_{i1}(t, x_i, v_i), f_{i2}(t, x_i, v_i), \cdots, f_{in}(t, x_i, v_i))^\top$, $i = 1, 2, \cdots, N$ are N nonidentical vector-valued continuous functions implying the inherent dynamics of agent i, $u_i(t)$ is the control input to be designed.

Since the nonlinear dynamics of N agents are different, the heterogenous multi-agent system is difficult to achieve consensus. Consider a virtual leader for multi-agent systems (1) is described by

$$\begin{cases} \dot{x}_0(t) = v_0(t), \\ \dot{v}_0(t) = f_0(t, x_0(t), v_0(t)), \end{cases} \tag{2}$$

where $x_0(t) \in R^n$ and $v_0(t) \in R^n$ are the position and velocity states of the virtual leader, respectively. $f_0 : R \times R^n \times R^n \to R^n$ is continuous.

Definition 1. The heterogeneous multi-agent system (1) is said to achieve quasi consensus with the leader if there exists a positive constant ε such that

$$\lim_{t\to\infty} \|x_i(t) - x_0(t)\| \le \varepsilon, \text{ and } \lim_{t\to\infty} \|v_i(t) - v_0(t)\| \le \varepsilon,$$

$i = 1, 2, \cdots, N$, hold for all initial values.

Consider the following pinning control algorithm to achieve quasi consensus between the heterogeneous multi-agent system (1) and the leader (2)

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = f_i(t, x_i(t), v_i(t)) + \alpha \sum\limits_{j=1, j\neq i}^{N} a_{ij}(x_j(t) - x_i(t)) \\ \qquad + \beta \sum\limits_{j=1, j\neq i}^{N} a_{ij}(v_j(t) - v_i(t)) + \\ \qquad d_i[(x_0(t) - x_i(t)) + (v_0(t) - v_i(t))], \end{cases} \tag{3}$$

where $\alpha > 0$ and $\beta > 0$ are the coupling strengths, the pinning control gain $d_i > 0$ if the ith node is selected to be pinned, otherwise $d_i = 0$.

Let $e_{i1}(t) = x_i(t) - x_0(t)$ and $e_{i2}(t) = v_i(t) - v_0(t), i = 1, 2, \cdots, N$. One can derive the following error systems

$$\begin{cases} \dot{e}_{i1}(t) = e_{i2}(t), \\ \dot{e}_{i2}(t) = [f_i(t, x_i(t), v_i(t)) - f_0(t, x_0(t), v_0(t))] \\ \qquad - \alpha \sum\limits_{j=1}^{N} l_{ij} e_{j1}(t) - \beta \sum\limits_{j=1}^{N} l_{ij} e_{j2}(t) \\ \qquad - d_i[e_{i1}(t) + e_{i2}(t)]. \end{cases} \tag{4}$$

Let $e_1(t) = [e_{11}(t), e_{21}(t), \cdots, e_{N1}(t)]^T$, $e_2(t) = [e_{12}(t), e_{22}(t), \cdots, e_{N2}(t)]^T$, $D = \mathtt{diag}\{d_1, d_2, \cdots, d_N\}$, and $F(t, x(t), x_0(t), v(t), v_0(t)) =$

$$\begin{pmatrix} f_1(t, x_1(t), v_1(t)) - f_0(t, x_0(t), v_0(t)) \\ f_2(t, x_2(t), v_2(t)) - f_0(t, x_0(t), v_0(t)) \\ \vdots \\ f_N(t, x_N(t), v_N(t)) - f_0(t, x_0(t), v_0(t)) \end{pmatrix}.$$

Rewrite (4) as

$$\begin{cases} \dot{e}_1(t) = e_2(t), \\ \dot{e}_2(t) = F(t, x(t), x_0(t), v(t), v_0(t)) \\ \qquad - ((\alpha\mathcal{L} + D) \otimes I_n)e_1(t) - ((\beta\mathcal{L} + D) \otimes I_n)e_2(t). \end{cases} \tag{5}$$

Throughout the rest of the paper, the following assumptions are needed.

Assumption 1. The isolate node of virtual leader moves in a bounded region consistently, that is, there exists a compact set $S \subset R^n \times R^n$ such that the isolate node will be always in S if it starts with $(x_0, v_0) \in S$.

Remark 1. The assumption 1 was adopted by some papers such as [5, 15] and can be satisfied by many well-knowing systems, such as Lorenz system, chaotic Chua system and so on.

Assumption 2. There exist positive numbers θ_{i1} and θ_{i2} such that

$$\|f_i(t, x, v) - f_i(t, y, u)\| \le \theta_{i1}\|x - y\| + \theta_{i2}\|v - u\|,$$

$\forall t \in R, x, y, v, u \in R^n, i = 1, 2, \cdots, N$.

In order to derive the main result of this paper, the following lemmas are need.

Lemma 1. (Schur complement, [16]). The following linear matrix inequality (LMI)

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0,$$

where $Q(x) = Q^T(x),\ R(x) = R^T(x)$, is equivalent to either of the following conditions:

(i) $Q(x) > 0,\ R(x) - S^T(x)Q^{-1}(x)S(x) > 0$;

(ii) $R(x) > 0,\ Q(x) - S(x)R^{-1}(x)S^T(x) > 0$.

Lemma 2. ([17]) Let matrix $A \in R^{N\times N}$ be symmetric. One has

$$\lambda_{\min}(A)x^T x \le x^T A x \le \lambda_{\max}(A)x^T x, \quad x \in R^N.$$

Consider a new graph $\overline{\mathcal{G}}$ contains n followers (related to graph $\mathcal{G}$) and the leader. $\overline{\mathcal{G}}$ is connected if at least one agent in each of its component of followers is connected with the leader. Throughout this paper, we always assume that $\overline{\mathcal{G}}$ is connected. One has the following lemma.

Lemma 3. ([18]) If graph $\overline{\mathcal{G}}$ is connected, then the symmetric matrix $\alpha\mathcal{L} + D$ and $\beta\mathcal{L} + D$ are positive definite.

3. Main result

In this section, we establish a sufficient condition for quasi consensus between the heterogeneous multi-agent system and the leader.

Theorem 1. Under Assumptions 1 and 2, the multi-agent systems can achieve quasi consensus with the leader if $\lambda_{\min}(\alpha\mathcal{L} + D) > 2\theta_1 + 1$ and $\lambda_{\min}(\beta\mathcal{L} + D) > 2\theta_2 + 2$, where $\theta_1 = \max\{\theta_{11}^2, \theta_{21}^2, \cdots, \theta_{N1}^2\}$, $\theta_2 = \max\{\theta_{12}^2, \theta_{22}^2, \cdots, \theta_{N2}^2\}$.

Proof. Let $e(t) = (e_1(t), e_2(t))^\top$. Consider the following Lyapunov functional candidate:

$$V(t) = \frac{1}{2}\, e^\top(t)(P \otimes I_n)e(t), \tag{6}$$

where $P = \begin{pmatrix} \alpha\mathcal{L} + \beta\mathcal{L} + 2D & I_N \\ I_N & I_N \end{pmatrix}$. By Lemma 1 and Lemma 3, we know that $P > 0$ is equivalent to $\alpha\mathcal{L} + \beta\mathcal{L} + 2D - I_N > 0$. Therefore, we can derive that $P > 0$ from the conditions of theorem.

Rewrite (6) as

$$V(t) = \frac{1}{2} e_1^\top(t)[(\alpha\mathcal{L} + \beta\mathcal{L} + 2D) \otimes I_n]e_1(t) + e_1^\top(t)e_2(t) + \frac{1}{2} e_2^\top(t)e_2(t) \tag{7}$$

Taking the derivative of $V(t)$ along the trajectories of (5) yields

$$\begin{aligned} \dot{V}(t) &= e_1^\top(t)[(\alpha\mathcal{L} + \beta\mathcal{L} + 2D) \otimes I_n]\dot{e}_1(t) + \\ &\quad \dot{e}_1^\top(t)e_2(t) + e_1^\top(t)\dot{e}_2(t) + e_2^\top(t)\dot{e}_2(t) \\ &= e_1^\top(t)[(\alpha\mathcal{L} + \beta\mathcal{L} + 2D) \otimes I_n]e_2(t) + e_2^\top(t)e_2(t) \\ &\quad + [e_1^\top(t) + e_2^\top(t)][F(t, x(t), x_0(t), v(t), v_0(t)) \\ &\quad - ((\alpha\mathcal{L} + D) \otimes I_n)e_1(t) - ((\beta\mathcal{L} + D) \otimes I_n)e_2(t)] \\ &= e_2^\top(t)e_2(t) - e_1^\top(t)(\alpha\mathcal{L} + D) \otimes I_n)e_1(t) \\ &\quad - e_2^\top(t)((\beta\mathcal{L} + D) \otimes I_n)e_2(t) \\ &\quad + [e_1^\top(t) + e_2^\top(t)]F(t, x(t), x_0(t), v(t), v_0(t)) \end{aligned} \tag{8}$$

Let $F(t, x(t), x_0(t), v(t), v_0(t)) = F_1 + F_2$, where

$$F_1 = \begin{pmatrix} f_1(t, x_1(t), v_1(t)) - f_1(t, x_0(t), v_0(t)) \\ f_2(t, x_2(t), v_2(t)) - f_2(t, x_0(t), v_0(t)) \\ \vdots \\ f_N(t, x_N(t), v_N(t)) - f_N(t, x_0(t), v_0(t)) \end{pmatrix},$$

and

$$F_2 = \begin{pmatrix} f_1(t, x_0(t), v_0(t)) - f_0(t, x_0(t), v_0(t)) \\ f_2(t, x_0(t), v_0(t)) - f_0(t, x_0(t), v_0(t)) \\ \vdots \\ f_N(t, x_0(t), v_0(t)) - f_0(t, x_0(t), v_0(t)) \end{pmatrix},$$

Therefore, one has

$$\begin{aligned} & [e_1^\top(t) + e_2^\top(t)]F_1 \\ \leq\ & \frac{1}{2}e_1^\top(t)e_1(t) + \frac{1}{2}e_2^\top(t)e_2(t) + F_1^\top F_1, \end{aligned} \quad (9)$$

and

$$\begin{aligned} & [e_1^\top(t) + e_2^\top(t)]F_2 \\ \leq\ & \frac{1}{2}e_1^\top(t)e_1(t) + \frac{1}{2}e_2^\top(t)e_2(t) + F_2^\top F_2, \end{aligned} \quad (10)$$

By Assumptions 1, 2, one derives

$$F_1^\top F_1 \leq 2\theta_1 e_1^\top(t)e_1(t) + 2\theta_2 e_2^\top(t)e_2(t), \quad (11)$$

and

$$F_2^\top F_2 = \sum_{i=1}^{N} \|f_i(t, x_0(t), v_0(t)) - f_0(t, x_0(t), v_0(t))\|^2 \leq M_0, \quad (12)$$

where $\theta_1 = \max\{\theta_{11}^2, \cdots, \theta_{N1}^2\}$, $\theta_2 = \max\{\theta_{12}^2, \cdots, \theta_{N2}^2\}$ and $M_0 = \max_{(x_0(0), v_0(0)) \in S} \{\sum_{i=1}^{N} \|f_i(t, x_0(t), v_0(t)) - g_0(t, x_0(t), v_0(t))\|^2\}$.

Substituting (9)-(12) into (8), one obtains

$$\begin{aligned} \dot{V}(t) \leq\ & M_0 - e_1^\top(t)(\alpha\mathcal{L} + D) \otimes I_n)e_1(t) \\ & - e_2^\top(t)((\beta\mathcal{L} + D) \otimes I_n)e_2(t) \\ & + (2\theta_1 + 1)e_1^\top(t)e_1(t) + (2\theta_2 + 2)e_2^\top(t)e_2(t) \\ =\ & -e^\top(t)(Q \otimes I_n)e(t) + M_0, \end{aligned} \quad (13)$$

where

$$Q = \begin{pmatrix} \alpha\mathcal{L} + D - (2\theta_1 + 1)I_N & O_N \\ O_N & \beta\mathcal{L} + D - (2\theta_2 + 2)I_N \end{pmatrix}.$$

By the conditions of Theorem 1, we can conclude that Q is positive definite. Therefore

$$V(t) \leq V(0)e^{-\lambda_{\max}(Q)t} + \frac{M_0}{\lambda_{\max}(Q)}, \quad (14)$$

together with Lemma 3 yields

$$\lim_{t \to \infty} \|e(t)\|^2 \leq \frac{M_0}{\lambda_{\max}(Q)\lambda_{\min}(P)} \quad (15)$$

According Definition 1, the heterogeneous multi-agent systems (1) can achieve quasi consensus with the leader (2).

4. Numerical results

Consider a heterogeneous multi-agent system with 3 nonidentical Chua's circuits of the nonlinear function f

$$f_i(t, x_i, v_i) = \begin{pmatrix} \gamma_i(-v_{i1} + v_{i2} - \iota(v_{i1})) \\ v_{i1} - v_{i2} + v_{i3} \\ -\delta_i v_{i2} \end{pmatrix}.$$

where $\iota(v_{i1}) = bv_{i1} + 0.5(a - b)(|v_{i1} + 1| - |v_{i1} - 1|)$, $\gamma_1 = 10, \gamma_2 = 20, \gamma_3 = 30, \delta_1 = 20, \delta_2 = 40, \delta_3 = 60$. Selecting $a = -4/3, b = -3/4$, the coupling strength $\alpha = \beta = 0.5$, pinning control gains $d_2 = 2$, and the coupling configuration matrix $\mathcal{A}$ is selected as $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Defining the virtual leader of the nonlinear dynamics

$$f_0(t, x_0, v_0) = \begin{pmatrix} \gamma_0(-v_{01} + v_{02} - \iota(v_{01})) \\ v_{01} - v_{02} + v_{03} \\ -\delta_0 v_{02} \end{pmatrix} - x_0.$$

where $\gamma_0 = 1, \delta_0 = 2$. Then by Theorem 1, quasi consensus between the heterogenous multi-agent system and the leader can be achieved.

Fig. 1 shows the trajectories of $x_{i1}(t)$ and v_{i1}, Fig. 2 shows the trajectories of $x_{i2}(t)$ and v_{i2}, and Fig. 3 shows the trajectories of $x_{i3}(t)$ and v_{i3}, $i = 0, 1, 2, 3$. The red star line is the trajectory of the leader.

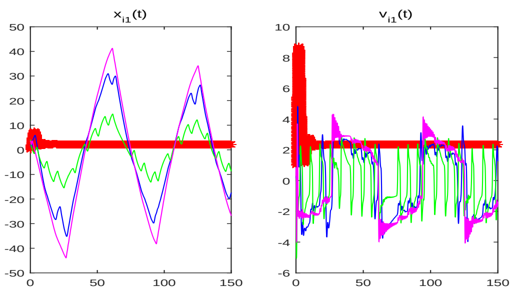

Figure 1: The trajectories of $x_{i1}(t)$ and v_{i1}, $i = 0, 1, 2, 3$.

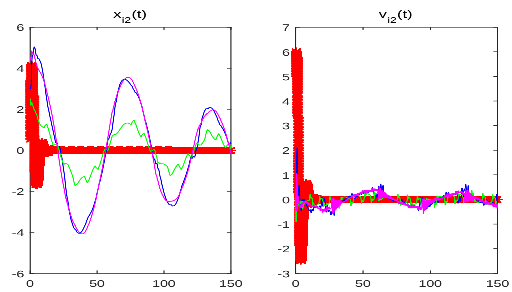

Figure 2: The trajectories of $x_{i2}(t)$ and v_{i2}, $i = 0, 1, 2, 3$.

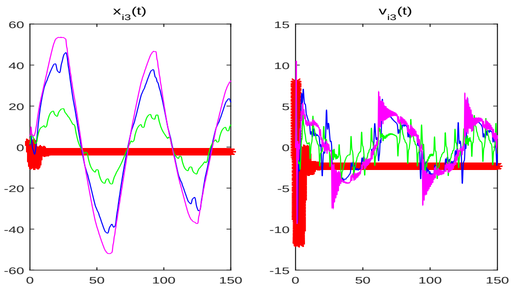

Figure 3: The trajectories of $x_{i2}(t)$ and v_{i2}, $i = 0, 1, 2, 3$.

5. Conclusions

This paper deals with leader-following consensus of heterogeneous multi-agent systems. By adopting a pinning controller, a sufficient condition for quasi consensus is obtained. In the future research, we will investigate the quasi consensus over the directed graph and in the general heterogeneous multi-agent systems.

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