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Synchronized States in Square Wave Oscillators Coupled by a Capacitor

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Abstract—We consider the synchronized states in a coupled square wave oscillators. First, we briefly show the behavior of the coupled square wave oscillators, and a hybrid return map can be realized by using the exact solution. Next, we make the return map the simplified it for the symmetrical property. From investigation of bifurcation diagram, we conclude that Saddle-Node and Border-Collision bifurcation influence the synchronization in these oscillators.

1. Introduction

The rhythmic motions have been analyzed by many scholars from a long time ago. One example of these is PET-bottle oscillator [1]. It is coupled two PET-bottles and observed the synchronized states between the downflow of water and the upflow of air [2]. M. Kohira et al. applied the PET-bottle oscillators and proposed the model equations by a coupled square wave oscillators [3]. These oscillators coupled by a resistor synchronize in the mode of in-phase. Their dynamics are described by four linear differential equations whose parameters are switched with threshold values. Recently, we also analyzed the oscillators coupled by a resistor, the cause of the synchronization in these oscillators was clarified [4]. On the other hand, the oscillators coupled by a capacitor synchronize in the mode of anti-phase. However, the coupled square wave oscillators by using a capacitor have not been established.

In this paper, we consider the synchronization in the square wave oscillators which are coupled by the capacitor. First, the exact solution is derived from the differential equations. Because of this, we can construct the return map. Next, we try to simplify the return map, because there is the symmetrical property in the trajectory of this system. Also, we show the return map of the synchronization, it is proven that these oscillators synchronize at antiphase. Moreover, we define the switching rate γ for various periodic and non-periodic trajectories by using the output pulses. Finally, we conclude that the Saddle-Node bifurcation and the Border-Collision bifurcation play an important role from investigation of one-parameter bifurcation diagram. Our method can be applied to the system which dynamics is described by four linear differential equations whose parameters are switched with threshold values.

2. Square Wave Oscillators Coupled by a Capacitor

Fig. 1shows the square wave oscillators coupled by a capacitor [3]. These oscillators are described as

$$\begin{cases} \dfrac{dv_1}{d\tau} = \dfrac{1}{1+2\delta}\left\{-(1+\delta)v_1 - \delta v_2 + (1+\delta)e_1 + \delta e_2\right\}, \\ \dfrac{dv_2}{d\tau} = \dfrac{1}{1+2\delta}\left\{-\delta v_1 - (1+\delta) + \delta e_1 + (1+\delta)e_2\right\}, \end{cases} \tag{1}$$

$$\begin{aligned} &v_{+1} = \alpha v_{01}, v_{+2} = \beta v_{02}, \\ &v_{01} = e\,\mathrm{sgn}(v_{+1} - v_1) = e\,\mathrm{sgn}(\alpha v_{01} - v_1), \\ &v_{02} = e\,\mathrm{sgn}(v_{+2} - v_2) = e\,\mathrm{sgn}(\beta v_{02} - v_2). \end{aligned} \tag{2}$$

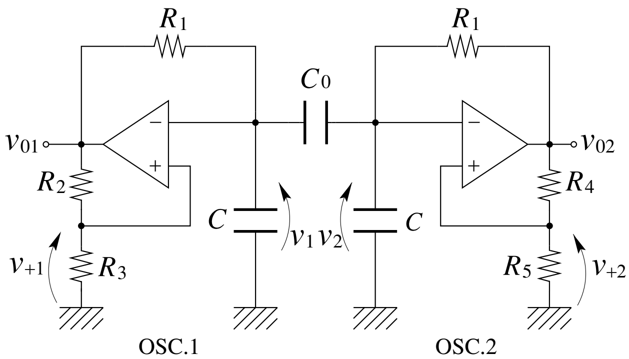

Figure 1: The coupled square wave oscillators.

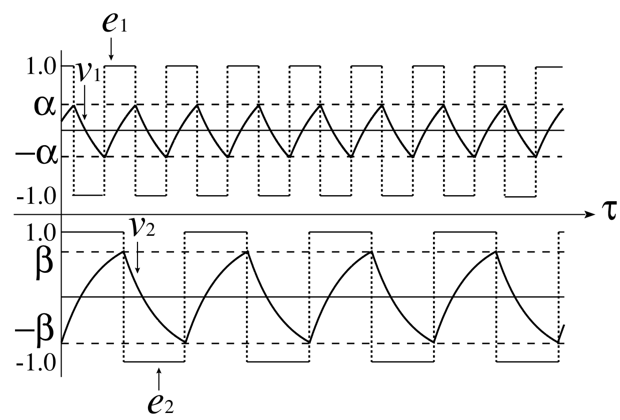

Figure 2: An example of the wave form. ($\alpha = 0.406$, $\beta = 0.7$, $\delta = 0.1$)

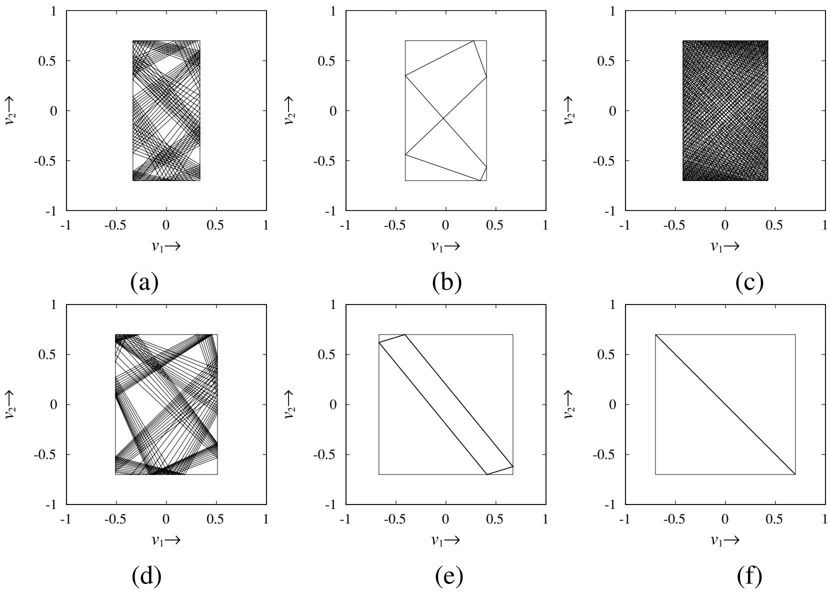

Figure 3: The phase portraits of some arbitrary parameters with $\beta = 0.7$, $\delta = 0.1$. (a) $\alpha = 0.345$, (b) $\alpha = 0.406$, (c) $= \alpha = 0.435$, (d) $\alpha = 0.52$, (e) $\alpha = 0.67$ and (f) $\alpha = 0.7$.

Equation (2) has the following relationship:

$$0 < \alpha < 1, 0 < \beta < 1. \tag{3}$$

Figure 2 shows an example of the wave form and Figs. 3 show the phase portraits with various values of α. Note that Fig. 2 corresponds Fig. 3(b). Here, when we transform and rescale for variables, Eq. (1) becomes the following differential equations:

$$\begin{cases} \dfrac{dx}{d\tau} + x = \dfrac{\sqrt{2}(e_1 + e_2)}{2}, \\ (1 + 2\delta)\dfrac{dy}{d\tau} + y = \dfrac{\sqrt{2}(e_1 - e_2)}{2}. \end{cases} \tag{4}$$

We define some objects as follows for the following analysis.

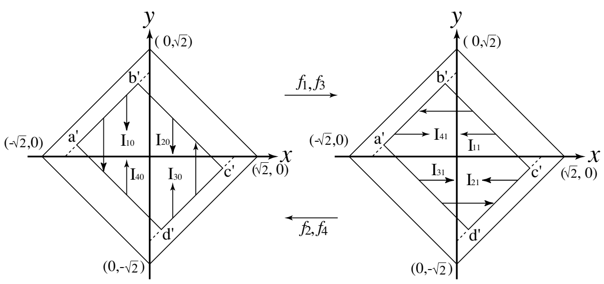

Figure 4: Some defined intervals.

$$\begin{aligned}
P_{11} &\equiv \{(x, y, e_1, e_2) \mid x \geq \frac{\sqrt{2}(\beta - \alpha)}{2}, y \geq x - \sqrt{2}\beta, \\ &\qquad e_1 = -1, e_2 = 1\}, \\
P_{12} &\equiv \{(x, y, e_1, e_2) \mid x \leq \frac{\sqrt{2}(\beta - \alpha)}{2}, y \geq -x - \sqrt{2}\alpha, \\ &\qquad e_1 = -1, e_2 = 1\}, \\
P_{21} &\equiv \{(x, y, e_1, e_2) \mid x \leq -\frac{\sqrt{2}(\alpha + \beta)}{2}, y \leq x + \sqrt{2}\beta, \\ &\qquad e_1 = -1, e_2 = -1\}, \\
P_{22} &\equiv \{(x, y, e_1, e_2) \mid x \leq -\frac{\sqrt{2}(\alpha + \beta)}{2}, y \geq -x - \sqrt{2}\alpha, \\ &\qquad e_1 = -1, e_2 = -1\}, \\
P_{31} &\equiv \{(x, y, e_1, e_2) \mid x \leq -\frac{\sqrt{2}(\beta - \alpha)}{2}, y \leq x + \sqrt{2}\beta, \\ &\qquad e_1 = 1, e_2 = -1\}, \\
P_{32} &\equiv \{(x, y, e_1, e_2) \mid x \geq -\frac{\sqrt{2}(\beta - \alpha)}{2}, y \leq -x + \sqrt{2}\alpha, \\ &\qquad e_1 = 1, e_2 = -1\}, \\
P_{41} &\equiv \{(x, y, e_1, e_2) \mid x \leq \frac{\sqrt{2}(\alpha + \beta)}{2}, y \leq -x + \sqrt{2}\alpha, \\ &\qquad e_1 = 1, e_2 = 1\}, \\
P_{42} &\equiv \{(x, y, e_1, e_2) \mid x \leq \frac{\sqrt{2}(\alpha + \beta)}{2}, y \geq x - \sqrt{2}\beta, \\ &\qquad e_1 = 1, e_2 = 1\},
\end{aligned} \tag{5}$$

$$\begin{cases} P_1 \equiv P_{11} \cap P_{12}, \; P_2 \equiv P_{21} \cap P_{22}, \\ P_3 \equiv P_{31} \cap P_{32}, \; P_4 \equiv P_{41} \cap P_{42}. \end{cases} \tag{6}$$

Equation (4) defines four linear differential equations on four half planes which are controlled by the output voltage of the operational amplifier with Eq. (6). In the following, we analyze Eq. (4) as the hybrid dynamical systems which are defined by these planes.

3. Return Map

The solution of the coupled square wave oscillators can be calculated analytically because Eq. (4) is a piecewise linear autonomous system. Therefore, we obtain a return map explicitly. Here, we show some intervals: I_{10}, I_{20}, I_{30}, I_{40}, I_{11}, I_{21}, I_{31}, I_{41} (See Fig. 4). So, the following mappings are defined:

$$\begin{array}{rcl} f_1 : (I_{10} \cup I_{20}) & \rightarrow & (I_{21} \cup I_{31}), \\ f_2 : (I_{11} \cup I_{21}) & \rightarrow & (I_{10} \cup I_{40}), \\ f_3 : (I_{30} \cup I_{40}) & \rightarrow & (I_{11} \cup I_{41}), \\ f_4 : (I_{31} \cup I_{41}) & \rightarrow & (I_{20} \cup I_{30}). \end{array} \tag{7}$$

Additionally, we define as follows:

$$\psi : f \rightarrow S^1, \quad (x, y) \rightarrow X', \tag{8}$$

where

$$X' = \begin{cases} \dfrac{(\alpha+\beta) + \sqrt{2}x}{2(\alpha+\beta)}, & x \in I_{10}, I_{20}, I_{11}, I_{41}, \\ \dfrac{3(\alpha+\beta) - \sqrt{2}x}{2(\alpha+\beta)}, & x \in I_{30}, I_{40}, I_{21}, I_{31}. \end{cases} \tag{9}$$

The following return map G can be defined by using Eq. (8).

$$\begin{array}{l} G \equiv \psi f \psi^{-1} \;\; : \;\; S^1 \rightarrow S^1, \\ \quad S^1 = \{X' \in \boldsymbol{R} \bmod 2\}. \end{array} \tag{10}$$

Thus, the dynamics of Eq. (4) can be interpreted as behavior of a discrete map written by:

$$X'_{n+1} = G(X'_n). \tag{11}$$

Now x is described by Eq. (9).

$$x = \begin{cases} \sqrt{2}(\alpha+\beta)\left(X' - \dfrac{1}{2}\right), & x \in I_{10}, I_{20}, I_{11}, I_{41}, \\ -\sqrt{2}(\alpha+\beta)\left(X' - \dfrac{3}{2}\right), & x \in I_{30}, I_{40}, I_{21}, I_{31}. \end{cases} \tag{12}$$

Figure 5 shows an example of G which corresponds to the parameters used in Fig. 3(b).

Next, we simplify the return map because the trajectory of this system has the symmetrical property [5]. For the simplification, we define the following:

$$Y_n \equiv e_1(n)e_2(n), \qquad (Y_n = 1 \text{ or } -1), \tag{13}$$

$$X = \begin{cases} X', & x \in I_{10}, I_{20}, I_{11}, I_{41}, \\ X' - 1, & x \in I_{30}, I_{40} I_{21}, I_{31}. \end{cases} \tag{14}$$

Finally, we can define the simplified return map as follows:

$$\begin{array}{l} F \equiv \psi f \psi^{-1} \;\; : \;\; S^1 \rightarrow S^1, \\ \quad S^1 = \{X \in \boldsymbol{R} \bmod 1\}. \end{array} \tag{15}$$

Here, we assume that a point X_p is called n-periodic point of F such that

$$F^n(X_p) = X_p, F^k(X_p) \neq X_p \; (k < n). \tag{16}$$

4. Analysis

Theorem 1: F has no fixed point.
Proof: The following transitions have to be considered.

case1: $I_{10} \rightarrow I_{21} \rightarrow I_{10}$ **case2:** $I_{20} \rightarrow I_{31} \rightarrow I_{20}$
case3: $I_{30} \rightarrow I_{41} \rightarrow I_{30}$ **case4:** $I_{40} \rightarrow I_{11} \rightarrow I_{40}$

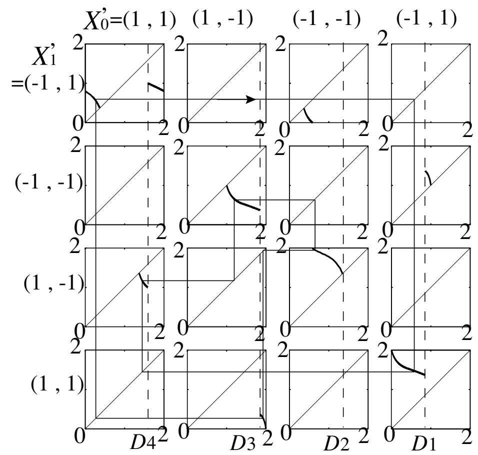

Figure 5: An example of the return map. ($\alpha = 0.406, \beta = 0.7, \delta = 0.1$)

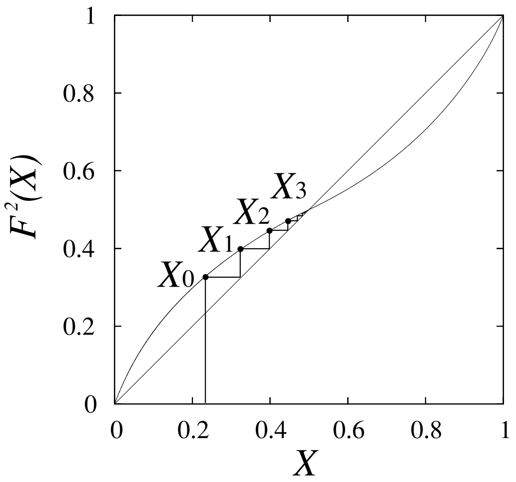

Figure 6: A return map of the synchronized states. ($\alpha = 0.7, \beta = 0.7, \delta = 0.1$)

In this paper, we show only **case1**. Let x_0 be initial point on I_{10}. x_0 is transformed to a point $x_1 \in I_{21}$ and x_1 is transformed to a point $x_2 \in I_{10}$. Because of $x_0 < x_1 < x_2$, F has no fixed point. Analogous manner is possible for the other cases.

Theorem 2: We assume $\alpha = \beta$, the coupled square wave oscillators synchronize in the mode of anti-phase.

Proof: When an initial point is located in $X_0 < 0.5$ or $X_0 > 0.5$, it is possible in the following cases ($\delta > 0$).

$$\lim_{n\to\infty} F^{2n}(X_0) \to 0.5. \tag{17}$$

It means that the stable fixed point has $X = 0.5$. That is to say, the coupled square wave oscillators synchronize at anti-phase (See Fig. 6).

Next, we analyze the coupled square wave oscillators with different parameter ($\alpha \neq \beta$). For characterizing the periodic trajectory, we show switching rate γ.Following properties are known [6].

1. γ exists and depends on the initial point X.
2. γ is rational if F has a stable periodic trajectory.

Property1: If $\gamma = n/m$, the switch of the output of the operational amplifier has $m+n$ switching. F has $m+n$ periodic point and $m + n$ denotes the number of line segmentations of a trajectory on v_1-v_2 plane.

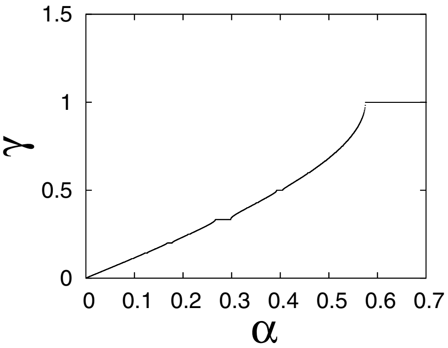

Figure 7: $\alpha - \gamma$ characteristic. $F^n(x)$, $n = 4000$, is used for the calculation of γ. ($\beta = 0.7, \delta = 0.2$)

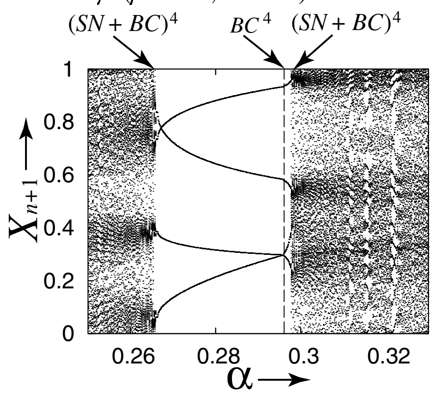

Figure 8: One-parameter bifurcation diagram when $\beta = 0.7, \delta = 0.2$.

Property2: If $\beta < \alpha$, γ is larger than 1, and if $\alpha < \beta$, γ is smaller than 1. In some cases, if γ is irrational, F has a periodic trajectory. Figure 7 shows the graph of the switching rate. This figure means that Eq. (4) has many periodic trajectories.

Finally, we analyze the bifurcation phenomena when $\gamma = 2/6$. Figure 8 shows 4-periodic and non-periodic solution. In this figure, SN corresponds to bifurcation point of Saddle-Node bifurcation and BC corresponds to bifurcation point of Border-Collision bifurcation. Here, Border-Collision bifurcation is bifurcation phenomena when the trajectory exceed the border. We conclude that Saddle-Node bifurcation and Border-Collision bifurcation essentially influence the synchronization of the coupled square wave oscillators.

5. Conclusions

We have analyzed the synchronized states in the coupled square wave oscillators. The systems are described by four linear differential equations, so that we have derived the return map explicitly. Furthermore the return map has been simplified for the symmetrical property. We also have discussed the bifurcation phenomena only when $\gamma = 2/6$. Detailed bifurcation analysis is the future works.

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