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Is Markov code superior to i.i.d. in communication systems?

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Abstract—We propose a CDMA system which allows each user's signal to have a frequency offset. Such a system can be regarded as a frequency dual of a chip-asynchronous DS/CDMA system. Time-domain Markovian codes reduce the variance of multiple-access interference (MAI) in chip-asynchronous CDMA systems, while in such a dual system, MAI can be reduced by frequency-domain Markov codes.

1. Introduction

The variance of multiple-access interference (MAI) in chip-asynchronous DS/CDMA systems with independent and identically distributed (i.i.d.) codes is smaller than that in chip-synchronous as well as symbol-synchronous DS/CDMA systems. The MAI is further reduced if we replace i.i.d. codes with Markovian spreading codes generated by a chaotic map [1, 2, 3]. This fact shows that a chaotic map is promising for a spread spectrum (SS) code generation. In this paper we consider a question: in which situation, Markov codes can outperform i.i.d. codes? This work is a preliminary study approaching to this question.

We propose a frequency dual of a chip-asynchronous DS/CDMA system. It is shown that Markov codes outperform i.i.d. ones in such a system, as well. Phase, timing, and frequency synchronization errors cause interference. Thus, a receiver which is robust to these errors is desirable. Therefore such a dual system which allows a random frequency offset is promissing. In a DS/CDMA system, symbol duration is divided into N chip intervals, where N is a spreading factor. In the dual system, on the other hand, frequency band is divided into several sub-bands. Therefore we refer to this system as frequency division (FD)-based CDMA, whereas DS system can be referred to as a time division (TD)-based system.

In order to investigate the condition for Markov codes to be superior to i.i.d., we compare the proposed system with orthogonal frequency division multiplex (OFDM) as well as frequency-hopping(FH)/CDMA. It is suggested that the interference caused by synchronization errors may be reduced by Markov codes, if a communication system is properly designed.

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2. A FD-based CDMA system

In chip-synchronous DS/CDMA systems with i.i.d. spreading codes, the normalised variance of MAI is 1, while it is reduced to 2/3 in chip-asynchronous systems. The variance of MAI is further reduced to $1/\sqrt{3}$ if i.i.d. codes are replaced by Markov codes. It is natural to ask: is the variance of MAI reduced by introducing a frequency offset, too? We give an affirmative answer to this question.

In DS-CDMA systems, one data duration is divided into several small sub-intervals with equal lengths, which are called chips. We consider a CDMA system which is a frequency dual of asynchronous DS/CDMA system, where one data bandwidth is divided into several frequency subbands and relative frequency offset between users are allowed. This system is regarded as a kind of multi-carrier CDMA system, because DS-CDMA is based on time division (TD), while its frequency dual version is based on frequency division.

In a chip-asynchronous DS/CDMA system, spread spectrum code signal is expressed as,

$$s_j(t) = \sum_{n=0}^{N-1} X_{n,j} u_{T_c}(t - nT_c), \tag{1}$$

where N is a spreading factor, T_c is a chip duration, $\boldsymbol{X}_j = (X_{0,j}, X_{1,j}, \ldots, X_{N-1,j})^T$ is a time domain spreading code sequence of the j-th user, and $u(t)$ is a rectangular chip waveform, i.e., $u_D(t) = 1$ for $-D/2 < t < D/2$ and $u_D(t) = 0$ otherwise. Then, the received signal with J users is

$$r(t) = \sum_{j=1}^{J} \sum_{p=-\infty}^{\infty} d_p^{(j)} s_j(t - \tau_j - pT) + n_0(t), \tag{2}$$

where $d_p^{(j)}$ is a data symbol of p-th period for j-th user, $T = NT_c$ is the data duration, and $n_0(t)$ is an additive white Gaussian noise. Two users' transmitted signals are illustrated in Fig. 1.

Suppose binary data symbols are $\{-1, 1\}$-valued, then two cases are considered, i.e., $d_p^{(j)} = d_{p+1}^{(j)}$ or $d_p^{(j)} = -d_{p+1}^{(j)}$. Correlation functions corresponding to these two cases are called even and odd functions, namely, $R_N^E(\ell; \boldsymbol{X}, \boldsymbol{Y}) = R_N^A(\ell; \boldsymbol{X}, \boldsymbol{Y}) + R_N^A(N - \ell; \boldsymbol{Y}, \boldsymbol{X})$ and $R_N^O(\ell; \boldsymbol{X}, \boldsymbol{Y}) = R_N^A(\ell; \boldsymbol{X}, \boldsymbol{Y}) - R_N^A(N - \ell; \boldsymbol{Y}, \boldsymbol{X})$, where

$$R_N^A(\ell; \boldsymbol{X}, \boldsymbol{Y}) = \sum_{n=0}^{N-1-\ell} X_n Y_{n+\ell} \quad (\ell \geq 0). \tag{3}$$

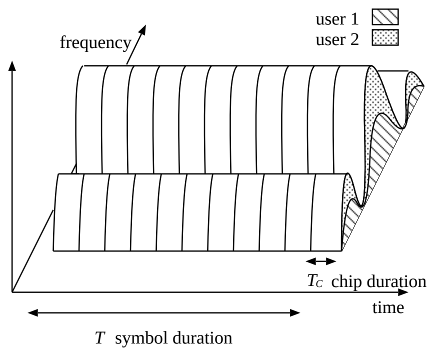

Figure 1: In DS/CDMA systems with a rectangular chip waveform, the spectrum of the transmitted signal is a sinc function. The users' signals are overlapped.

is the Pursley's aperiodic cross-correlation function between $\boldsymbol{X}$ and $\boldsymbol{Y}$ [4]. As in [5], we assume that for some positive integer m, $\ell_{ij} \in \{0, 1, \ldots, N-1\}$, and $k_{ij} \in \{0, 1, \ldots, m-1\}$, the relative time delay is expressed as

$$\tau_i - \tau_j = \left(\ell_{ij} + \frac{k_{ij}}{m}\right) T_c \tag{4}$$

The system is said to be *synchronous* if $\ell_{ij} = k_{ij} = 0$ for all i, j, *chip-synchronous* if $k_{ij} = 0$ but $\ell_{ij} \neq 0$, and *chip-asynchronous* if $k_{ij} \neq 0$. The multiple-access interference (MAI) is defined as

$$I_{J,p}^{(i)} = \sum_{j=1, j\neq i}^{J} \left\{ \frac{d_p^{(j)} + d_{p+1}^{(j)}}{2} \frac{1}{m} R_{mN}^{E}(\ell_{ij} m + k_{ij}; \boldsymbol{X}_{i,\mathrm{up}}, \boldsymbol{X}_{j,\mathrm{up}}) + \frac{d_p^{(j)} - d_{p+1}^{(j)}}{2} \frac{1}{m} R_{mN}^{O}(\ell_{ij} m + k_{ij}; \boldsymbol{X}_{i,\mathrm{up}}, \boldsymbol{X}_{j,\mathrm{up}}) \right\}, \tag{5}$$

where $\boldsymbol{X}_{\mathrm{up}}$ is an up-sampled sequence by a factor of m, defined as

$$\boldsymbol{X}_{\mathrm{up}} = \{\underbrace{X_0, \ldots X_0}_{m}, \underbrace{X_1, \ldots X_1}_{m}, \ldots, \underbrace{X_{N-1}, \ldots X_{N-1}}_{m}\}. \tag{6}$$

Let us consider a CDMA system which is a frequency dual of (1) and (2). The Fourier transform of these equations are, respectively, $\hat{s}_j(f) = \sum_{n=0}^{N-1} X_{n,j} T_c \mathrm{sinc}(T_c f) e^{\mathrm{j}\pi f(1-2n)T_c}$ and $\hat{r}(f) = \sum_{j=1}^{K} \sum_{p=-\infty}^{\infty} d_p^{(j)} \hat{s}_j(f) e^{-\mathrm{j}2\pi f(\tau_j + pT)} + \hat{n}_0(f)$, where $\mathrm{sinc}(t) = \sin(\pi t)/(\pi t)$. Replacing the time domain spreading codes $\boldsymbol{X}_j$, the frequency variable f, the data duration T, the chip duration T_c, the spreading factor N, and the time delay τ_j, respectively, by a frequency domain spreading code $\boldsymbol{X}'_j$, a time variable t, a bandwidth W, a chip-bandwidth W_c, a spreading factor in frequency domain $M = W/W_c$, and a frequency offset ν_j, we obtain:

$$s'_j(t) = \sum_{n=0}^{M-1} X'_{n,j} W_c \mathrm{sinc}(W_c t) e^{-\mathrm{j}2\pi n W_c t}, \tag{7}$$

$$r'(t) = \sum_{j=1}^{K} \sum_{p=-\infty}^{\infty} d_p^{(j)} s'_j(t) e^{-\mathrm{j}2\pi(\nu_j + pW)t} + n_0(t), \tag{8}$$

where the prime sign $(\cdot)'$ is used to express a frequency version of $(\cdot)$. Such a FD-based CDMA system is illustrated in Fig. 2.

Remark 1: Obviously, the frequency spectrum of Eq.(7) is expressed in the same form as the time domain expression of the transmitted signal in DS/CDMA systems. Such a FD-based CDMA system can be regarded as a kind of OFDM system which allows frequency offset ν_j, whereas standard OFDM systems do not allow it [6].

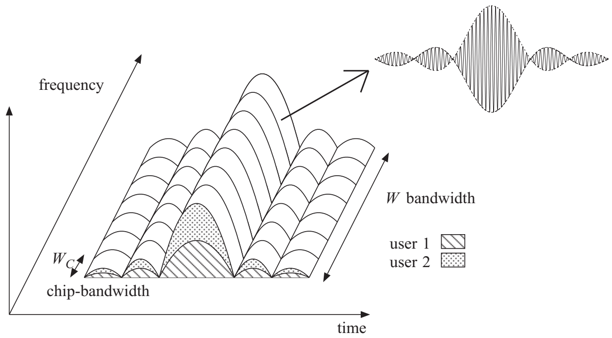

Figure 2: Frequency division (FD)-based CDMA system with sinc waveforms. It is permitted for each user's transmitted signal to have frequency offsets.

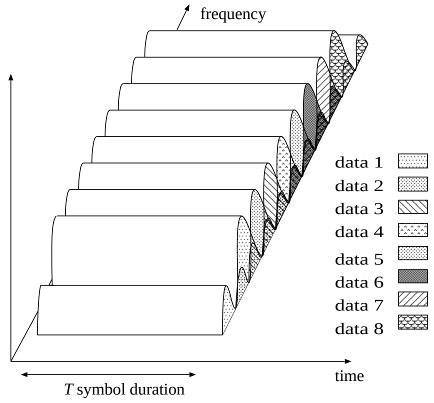

Figure 3: An OFDM system with rectangular waveforms. Sub-carriers are overlapping each other. In order to keep orthogonality between subcarriers, a frequency offset is not allowed.

An OFDM system is illustrated in Fig. 3, where several data are transmitted in parallel with the same number of subcarriers. The spectrums of adjacent subcarriers are overlapping each other.

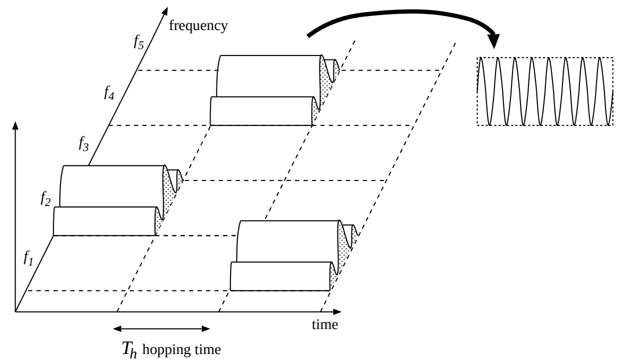

Figure 4: A FH/CDMA system. A rectangular baseband signal is modulated with a sinusoidal signal at frequency f_i. In FH systems, only one carrier is used in one hopping duration, whereas in the proposed system, a data signal multiplied by frequency domain spreading codes uses all carriers simultaneously.

Remark 2: The proposed system is similar to frequency hopping (FH)/CDMA but different from it in the following sense. In FH/CDMA, each user has a hopping pattern (Fig. 4), which determines a carrier frequency, by which the user's baseband signal is modulated. The carrier frequency changes at every hopping duration T_h. In a FH system, one user occupies only one carrier within a hopping duration. Therefore, hopping pattern of different users must be disjoint. If two users share the same frequency at the same time, bit error rate increases rapidly. On the other hand, in the FD-based CDMA system, data signal is multiplied by frequency-domain spreading codes, and every user can utilize full of the sub-carriers. Then transmitted signals are overlapping each other, which causes MAI. However, as in DS/CDMA systems, BER does not increase rapidly. Due to the property called 'graceful degradation', BER increases gradually.

Table 1: Correspondence Table-I

	Time Division	Frequency Division
rect. wave	DS/CDMA(Fig.1)	OFDM(Fig.3), FH-CDMA(Fig.4)
sinc wave		FD-based CDMA(Fig.2)
Gauss wave	UWB	

Table 2: Correspondence Table-II

	Time Division	Frequency Division
no spreading		OFDM
random spreading A	DS-CDMA	FD-based CDMA
random spreading B	UWB	FH-CDMA

The differences between the proposed method and existing ones are summerised in Table 1 and 2. For simplicity, only typical waveforms are listed in Table 1, although other Nyquist waveforms, e.g. raised cosine, are widely used. Ultra wideband (UWB) based on pulse position modulation (PPM) uses an impulsive Gaussian pulse, or Gausian monocycle [8]. Spreading codes are categorized into two types. The first type (random spreading A) is the one used in DS/CDMA and FD-based CDMA systems, i.e. it is a sequence of $\{+1, -1\}$-valued random variables. The second (random spreading B) is used in FH/CDMA system, i.e. it is a sequence of vectors of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the possition of '1' is randomly selected. Note that we have given intuitive illustrations in Figs. 1-4 and they were not well-established time-frequency representations, such as spectrogram (the square magnitude of short time Fourier transform) nor the Wigner distribution. We believe such intuitive illustrations are more appropriate to show the differences of the methods.

For frequency domain spreading codes, we have

$$R_M^A(\ell'; \boldsymbol{X}', \boldsymbol{Y}') = \sum_{n=0}^{M-1-\ell'} X'_n Y'_{n+\ell'} \quad (\ell' \geq 0). \tag{9}$$

We replace (4) by

$$\nu_i - \nu_j = \left(\ell'_{ij} + \frac{k'_{ij}}{m}\right) W_c. \tag{10}$$

The system is said to be *frequency synchronous* if $\ell'_{ij} = k'_{ij} = 0$, *frequency chip-synchronous* if $\ell'_{ij} \neq 0$ and $k'_{ij} = 0$, and *frequency chip-asynchronous* if $k'_{ij} \neq 0$. Then, the duality of such a system with DS/CDMA implies that the variance of MAI in this system is expressed in the same form of Eq. (5), i.e.,

$$\begin{aligned} I_{J,p}^{(i)}{}' = \sum_{j=1, j\neq i}^{J} \Bigg\{ & \frac{d_p^{(j)} + d_{p+1}^{(j)}}{2} \frac{1}{m} R_{mM}^E(\ell'_{ij} m + k'_{ij}; \boldsymbol{X}'_{i,\mathrm{up}}, \boldsymbol{X}'_{j,\mathrm{up}}) \\ & + \frac{d_p^{(j)} - d_{p+1}^{(j)}}{2} \frac{1}{m} R_{mM}^O(\ell'_{ij} m + k'_{ij}; \boldsymbol{X}'_{i,\mathrm{up}}, \boldsymbol{X}'_{j,\mathrm{up}}) \Bigg\}, \end{aligned} \tag{11}$$

Therefore the variance of MAI in the dual system is reduced in exactly the same way of chip-asynchronous DS/CDMA systems, except that the time delay τ_i is replaced by a frequency offset ν_i.

Assume that $d_p^{(i)}$ $(j \neq i)$ are independent on $d_p^{(j)}$. Then without loss of generality, it suffices to consider the MAI of two-user system. We replace $\boldsymbol{X}'_{i,\mathrm{up}}$, $\boldsymbol{X}'_{j,\mathrm{up}}$, ℓ'_{ij} and k'_{ij}, respectively, by $\boldsymbol{X}'_{\mathrm{up}}$, $\boldsymbol{Y}'_{\mathrm{up}}$, ℓ' and k'. As in DS/CDMA case, for a TD-based CDMA we have

Lemma 1: For any $\boldsymbol{X}'_{\mathrm{up}}$ and $\boldsymbol{Y}'_{\mathrm{up}}$, the MAI of two-user system satisfies

$$\begin{aligned} \left(I_{2,p}^{(i)}{}'\right)^2 = & \left(\frac{d_p^{(j)} + d_{p+1}^{(j)}}{2}\right)^2 \frac{1}{m^2} R_{mM}^E(\ell' m + k'; \boldsymbol{X}'_{\mathrm{up}}, \boldsymbol{Y}'_{\mathrm{up}})^2 \\ & + \left(\frac{d_p^{(j)} - d_{p+1}^{(j)}}{2}\right)^2 \frac{1}{m^2} R_{mM}^O(\ell' m + k'; \boldsymbol{X}'_{\mathrm{up}}, \boldsymbol{Y}'_{\mathrm{up}})^2 \end{aligned} \tag{12}$$

We have assumed a sinc waveform with rectangular spectrum, which implies that $\frac{1}{m}R^{E/O}_{mM}(\ell' m+k'; \boldsymbol{X}'_{\rm up}, \boldsymbol{X}'_{\rm up})$ has the relation

$$\frac{1}{m}R^{E/O}_{mM}(\ell' m + k'; \boldsymbol{X}'_{\rm up}, \boldsymbol{X}'_{\rm up}) = \left(1 - \frac{k'}{m}\right) R^{E/O}_{M}(\ell'; \boldsymbol{X}', \boldsymbol{Y}') + \frac{k'}{m} R^{E/O}_{M}(\ell' + 1; \boldsymbol{X}', \boldsymbol{Y}'). \quad (13)$$

Let $D^{(j)}$ be a $\{-1, 1\}$-valued random variable for $d^{(j)}_p$. Eqs. (11) and (13) together with the relation $\mathbf{E}[D^{(j)}_p D^{(j)}_{p+1}] = 0$ gives

$$\mathbf{E}_{D^{(j)}}\left[\mathbf{E}_{X^{(f)}Y^{(f)}}\left[\left(\frac{1}{\sqrt{M}} I^{(i)\prime}_{2,p}\right)^2\right]\right] = \left(1 - \frac{k'}{m}\right)^2 \mathcal{E}'_+(\ell') + \frac{k'^2}{m^2}\mathcal{E}'_+(\ell' + 1) + 2\left(1 - \frac{k'}{m}\right)\frac{k'}{m}\mathcal{F}'_+(\ell'), \quad (14)$$

where $\mathcal{E}'_+(\ell') = \frac{1}{2}(\mathcal{E}'^E(\ell') + \mathcal{E}'^O(\ell'))$ $\mathcal{F}'_+(\ell') = \frac{1}{2}(\mathcal{F}'^E(\ell') + \mathcal{F}'^O(\ell'))$ and

$$\mathcal{E}'^{E/O}(\ell') = \frac{1}{M}\mathbf{E}_{X'Y'}[R^{E/O}_M(\ell'; \boldsymbol{X}', \boldsymbol{Y}')^2] \quad (15)$$

$$\mathcal{F}'^{E/O}(\ell') = \frac{1}{M}\mathbf{E}_{X'Y'}[R^{E/O}_M(\ell'; \boldsymbol{X}', \boldsymbol{Y}') \cdot R^{E/O}_M(\ell' + 1; \boldsymbol{X}', \boldsymbol{Y}')] \quad (16)$$

Remark 3: Let K be a random variable for k' taking values in $\{0, 1, \ldots, m - 1\}$ with its probability $\Pr\{K = k'\} = 1/m$. Then for $m \gg 1$, we have $\mathbf{E}_K[(1 - K/m)^2] = 1/3$, $\mathbf{E}_K[(K/m)^2] = 1/3$ and $\mathbf{E}_K[(1 - K/m)K/m] = 1/6$. Then,

$$\mathbf{E}_K\left[\mathbf{E}_{D^{(j)}}\left[\mathbf{E}_{X^{(f)}Y^{(f)}}\left[(\tfrac{1}{\sqrt{M}} I^{(i)}_{2,p})^2\right]\right]\right] = \frac{1}{3}(\mathcal{E}'_+(\ell') + \mathcal{E}'_+(\ell' + 1) + \mathcal{F}'_+(\ell')) \quad (m \gg 1). \quad (17)$$

This implies the variance of MAI is reduced by negative $\mathcal{F}_+(\ell)$. This phenomenon is the same as the *antithetic variates* method in a variance reduction technique [7].

A negative $\mathcal{F}'(\ell')$ can be realized by Markov codes. $\mathcal{E}'(\ell')$ and $\mathcal{F}'(\ell')$ are, respectively, same as $\mathcal{E}(\ell)$ and $\mathcal{F}(\ell)$, except that variables are replaced according to Table 1. Thus, the evaluation of them for FD-based CDMA is the same as TD-based CDMA. Assume $X'_0 \to X'_1 \to \ldots, X'_{N-1}$ forms a Markov chain with a state space $\{+1, -1\}$. Let the eigenvalue of the transition probability matrix of the Markov chain be $-1 < \lambda < 1$. Then,

$$\mathcal{E}'_+(\ell') = \frac{1 + \lambda^2}{1 - \lambda^2}, \quad \mathcal{F}'_+(\ell') = \frac{2\lambda}{1 - \lambda^2}. \quad (18)$$

Therefore $\mathcal{F}_+(\ell')$ is negative for $\lambda < 0$. The optimum λ is $-2 + \sqrt{3}$ [3].

In ordinary OFDM systems, frequency offset $\nu_j \neq 0$ is not allowed. In such a case,

$$\mathbf{E}_{D^{(j)}}\left[\mathbf{E}_{X^{(f)}Y^{(f)}}\left[(\tfrac{1}{\sqrt{M}} I^{(i)}_{2,p})^2\right]\right] = \frac{1 + \lambda^2}{1 - \lambda^2} \quad (m = 1). \quad (19)$$

Hence, $\lambda = 0$ is optimum, which implies we cannot reduce the variance of MAI if $\nu_j = 0$.

The phase locked loop (PLL) circuit is a fundamental component of communications, which is needed for a tracking of frequency as well as phase synchronisations. It is desirable if a fine tuning of such synchronizations is not needed. For this purpose, the receiver must be designed to be robust to synchronizatoin errors. We have shown that the variance of MAI of frequency chip-asynchronous system is smaller than that of frequency chip-synchronous one. This implies that frequency synchronisation-free CDMA system is promising.

3. Concluding Remarks

This is a preliminary work for investigating the superiority of Markov codes over i.i.d. ones in communication systems. We propose a FD-based CDMA system which can be regarded as a frequency dual of chip-asynchronous DS/CDMA systems. In such a system, the variance of MAI is reduced, if the transmitted signal have a random frequency offset. The MAI becomes even smaller if we employ Markov codes instead of i.i.d. codes. This result strongly suggests that there are still other communication systems which allows both of time and frequency offsets, where Markov codes can make the system more resitant to the synchronization errors.

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