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Robust Stability for Complex-Valued Delayed Networks

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Abstract—This paper investigates the global robust stability problem for the complex-valued networks with constant delay, where the system matrix parameters are time-varying within the given intervals. Based on the Lyapunov functional method and some properties of the norm of matrices, sufficient criteria are presented to ascertain the global stability of the uncertain networks. The activation functions discussed here are no longer required to be derivable. What is more, the results established depend not only on the lower bound but also the upper bound information of the uncertain matrices, whereas the existed related criteria just utilize the lower bound information of the uncertain self-feedback matrices. One numerical example is also provided to illustrate the effectiveness of the obtained results.

1. Introduction

In the past few years, due to the extensive applications such as classification of signal processing, pattern recognition, and associative memory and so on, the recurrent networks have been widely studied, see [1, 2] for example. It is well known that time delays often occur in signal transmission among neurons in the electronic implementation of networks, which would influence the dynamics of the networks. Recently, dynamics of complex-valued neural networks with time delays have been widely studied, see [3, 4] and the references cited therein.

As is known to us, the activation functions play an important role in the dynamics of neural systems. However, according to Liouville's theorem, every bounded entire function must be a constant function in the complex domain. Therefore, it is a big challenge to choose appropriate activation functions for the complex-valued networks. Recently, different kinds of complex-valued activation functions have been proposed. It should be mentioned that in [5], the derivatives of the activation functions are supposed to exist and be continuous, however, in this paper, both the real part and imaginary part of the activation functions are no longer required to be derivable.

In practical implementation of neural systems, the values of the weight coefficients are subject to uncertainties. Robust stability results could be seen in references [6, 7]. To the best knowledge of the authors, there are not enough papers which are concerned about the robust stability of uncertain complex-valued networks with delays. Motivated by the above discussions, the aim of this paper is to study the robust stability of complex-valued recurrent neural networks with interval parameter uncertainties and time delays.

2. Preliminaries

Consider the following complex-valued recurrent neural networks with time delay as follows:

$$\dot{z}(t) = -Cz(t) + Af(z(t)) + Bf(z(t-\tau)) + L, \tag{1}$$

where $z(t) = (z_1(t), \ldots, z_n(t))^T \in \mathbb{C}^n$ is the state vector of the neural networks with n neurons at time t, $C = \text{diag}\{c_1, \cdots, c_n\} \in \mathbb{R}^{n\times n}$ with $c_k > 0$ $(k = 1, \ldots, n)$ is the self-feedback connection weight matrix, $A = (a_{kj})_{n\times n} \in \mathbb{C}^{n\times n}$ and $B = (b_{kj})_{n\times n} \in \mathbb{C}^{n\times n}$ are, respectively, the connection weight matrix and the delayed connection weight matrix. τ is the constant time delay and $L = (l_1, \ldots, l_n)^T \in \mathbb{C}^n$ is the external input vector. $f(z(t)) = (f_1(z_1(t)), \ldots, f_n(z_n(t)))^T : \mathbb{C}^n \to \mathbb{C}^n$ denotes the vector-valued activation function, which satisfies the condition given bellow.

Assumption 1. Let $z = z_1 + iz_2$ with $z_1, z_2 \in \mathbb{R}$. $f_k(z)$ can be expressed by its real and imaginary parts with

$$f_k(z) = f_k^R(z_1) + if_k^I(z_2),$$

where $k = 1, 2, \cdots, n$; $f_k^R(\cdot)$, $f_k^I(\cdot)$: $\mathbb{R} \to \mathbb{R}$ are bounded and monotonically nondecreasing which satisfy the following Lipschitz continuous condition

$$|f_k^R(z_1) - f_k^R(\widehat{z}_1)| \le r_k|z_1 - \widehat{z}_1|,$$
$$|f_k^I(z_2) - f_k^I(\widehat{z}_2)| \le s_k|z_2 - \widehat{z}_2|;$$

in which r_k, s_k are known constants, and z_1, z_2, $\widehat{z}_1$, $\widehat{z}_2$ are any numbers in $\mathbb{R}$.

With the above Assumption, if we denote $z(t) = x(t) + iy(t)$ with $x(t), y(t) \in \mathbb{R}^n$, then the complex-valued recurrent neural network (1) can be rewritten as follows:

$$\dot{v}(t) = -\widetilde{C}v(t) + \widetilde{A}g(v(t)) + \widetilde{B}g(v(t-\tau)) + \zeta, \tag{2}$$

where $g(v(t)) = ((f^R(x(t)))^T, (f^I(y(t)))^T)^T$, $v(t) = (x^T(t), y^T(t))^T$, $\zeta = ((L^R)^T, (L^I)^T)^T$, $\widetilde{C} = \text{diag}\{C, C\}$,

$$\widetilde{A} = \begin{pmatrix} A^R & -A^I \\ A^I & A^R \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} B^R & -B^I \\ B^I & B^R \end{pmatrix}.$$

It is well known that the bounded activation functions could always guarantee the existence of the equilibrium points for system (2). Assume that $v^* = (v_1^*, v_2^*, \dots, v_{2n}^*)$ is an equilibrium point of (2). In order to simplify the proofs, by setting $u(t) = v(t) - v^*$, system (2) can be transformed into the following form:

$$\dot{u}(t) = -\widetilde{C}u(t) + \widetilde{A}\phi(u(t)) + \widetilde{B}\phi(u(t-\tau)), \tag{3}$$

where $\phi(u(t)) = g(u(t)+v^*) - g(v^*)$. According to Assumption 1, it is easy to know that the function $\phi(\cdot)$ is monotonically nondecreasing and

$$|\phi_k(u_k)| \le h_k|u_k|,\ \forall u_k \in \mathbb{R},\ k = 1, 2, \dots, 2n \tag{4}$$

where $h_k = r_k$ $(k = 1, 2, \dots, n)$ and $h_k = s_k$ $(k = n+1, n+2, \dots, 2n)$.

Lemma 1.[8] For matrix $E = (e_{kj})_{n\times n} \in [\underline{E}, \overline{E}]$, i.e., $\underline{e}_{kj} \le e_{kj} \le \overline{e}_{kj}$ with $\underline{E} = (\underline{e}_{kj})_{n\times n}$ and $\overline{E} = (\overline{e}_{kj})_{n\times n}$, we have $\|E\|_2 \le \|E^*\|_2 + \|E_*\|_2$, where $E^* = (\overline{E}+\underline{E})/2$, $E_* = (\overline{E}-\underline{E})/2$ and $\|E\|_2$ is the induced 2-norm of matrix E.

3. Main results

In this section, by utilizing the Lyapunov functional method and some inequalities analysis, we give, respectively, the result for the deterministic networks and the result for uncertain networks.

Theorem 1. Suppose that the Assumption 1 is satisfied, then the complex-valued neural network (1) is globally asymptotically stable if there exist two diagonal matrices $P = \mathrm{diag}\{p_1, p_2, \dots, p_{2n}\} > 0$ and $D = \mathrm{diag}\{d_1, d_2, \dots, d_{2n}\} > 0$ such that

$$\begin{aligned}\Xi = P\widetilde{C} + \widetilde{C}P - P\widetilde{A}\widetilde{C}^{-1}\widetilde{A}^T P\\ -P\widetilde{B}D^{-1}\widetilde{B}^T P - (\widetilde{C} + D)H^2 > 0.\end{aligned} \tag{5}$$

where $H = \mathrm{diag}\{h_1, h_2, \dots, h_{2n}\}$.

Proof. First, the uniqueness of the equilibrium point will be proved. Consider the equation as follows:

$$\widetilde{C}u^* - \widetilde{A}\phi(u^*) - \widetilde{B}\phi(u^*) = 0. \tag{6}$$

It is easy to know that if $u^* = 0$, then equation (6) holds, i.e., $u^* = 0$ is an equilibrium of (3). Now, let $u^* \ne 0$, multiply both sides of (6) by $2(u^*)^T P$, one gets that

$$\begin{aligned}&2(u^*)^T P\widetilde{C}u^* = 2(u^*)^T P\widetilde{A}\phi(u^*) + 2(u^*)^T P\widetilde{B}\phi(u^*)\\ &\le (u^*)^T P\widetilde{A}\widetilde{C}^{-1}\widetilde{A}^T Pu^* + (u^*)^T P\widetilde{B}D^{-1}\times\\ &\quad \widetilde{B}^T Pu^* + \phi^T(u^*)(\widetilde{C} + D)\phi(u^*).\end{aligned} \tag{7}$$

It follows from (4) that

$$\phi^T(u)\phi(u) \le u^T H^2 u,\ \forall u \in \mathbb{R} \tag{8}$$

from which, one could obtain that

$$\phi^T(u^*)(\widetilde{C} + D)\phi(u^*) \le (u^*)^T(\widetilde{C} + D)H^2u^*. \tag{9}$$

Substituting inequality (9) into (7) yields that

$$(u^*)^T\Xi u^* \le 0. \tag{10}$$

From condition (5), one has that for $u^* \ne 0$

$$(u^*)^T\Xi u^* > 0, \tag{11}$$

which contradicts with (10) and hence implies that the equilibrium point $u^* = 0$ is unique. This means the equilibrium point of system (1) is also unique.

Second, we shall prove that system (3) is globally asymptotically stable. Consider the following Lyapunov functional candidate:

$$V(u(t)) = u^T(t)Pu(t) + \int_{t-\tau}^{t} \phi^T(u(s))D\phi(u(s))ds. \tag{12}$$

Calculating the time derivative of $V(u(t))$ along the trajectories of (3), it can be obtained that

$$\begin{aligned}\dot{V}(u(t)) \le& -2u^T(t)P\widetilde{C}u(t) + u^T(t)P\widetilde{A}\widetilde{C}^{-1}\widetilde{A}^T Pu(t)\\ &+ \phi^T(u(t))\widetilde{C}\phi(u(t)) + \phi^T(u(t))D\phi(u(t))\\ &+ u^T(t)P\widetilde{B}D^{-1}B^T Pu(t)\\ \le& -u(t)^T\Big(P\widetilde{C} + \widetilde{C}P - P\widetilde{A}\widetilde{C}^{-1}\widetilde{A}^T P\\ &-P\widetilde{B}D^{-1}\widetilde{B}^T P - (\widetilde{C} + D)H^2\Big)u(t).\end{aligned}$$

Since $P\widetilde{C} + \widetilde{C}P - P\widetilde{A}\widetilde{C}^{-1}\widetilde{A}^T P - P\widetilde{B}D^{-1}\widetilde{B}^T P - (\widetilde{C} + D)H^2$ is positive definite, hence $\dot{V}(u(t)) < 0$ holds for all $u(t) \ne 0$. Therefore, network (3) or equivalently system (1) is globally asymptotically stable. The proof is complete. □

As discussed in the Introduction, system matrices are always indeterministic when describing the practical physical plant due to various reasons. In this paper, it is assumed that the parameters C, A and B of the complex-valued recurrent neural networks (1) are uncertain and bounded. To be specific, $C \in C_r$, $A \in A_r$ and $B \in B_r$, where matrices sets C_r, A_r and B_r are described as follows:

$$\begin{cases} C_r := \{C = \mathrm{diag}\{c_1, \dots, c_n\} |\ 0 < \underline{c}_k \le \\ \qquad c_k \le \overline{c}_k\},\\ A_r := \{A = (a_{kj})_{n\times n} |\ \underline{a}_{kj}^R \le a_{kj}^R \le \overline{a}_{kj}^R,\\ \qquad \underline{a}_{kj}^I \le a_{kj}^I \le \overline{a}_{kj}^I\},\\ B_r := \{B = (b_{kj})_{n\times n} |\ \underline{b}_{kj}^R \le b_{kj}^R \le \overline{b}_{kj}^R,\\ \qquad \underline{b}_{kj}^I \le b_{kj}^I \le \overline{b}_{kj}^I;\ k, j = 1, 2, \dots, n\}, \end{cases} \tag{13}$$

in which $\underline{C} = \mathrm{diag}\{\underline{c}_1, \dots, \underline{c}_n\}$, $\overline{C} = \mathrm{diag}\{\overline{c}_1, \dots, \overline{c}_n\}$, $\underline{A} = (\underline{a}_{kj})_{n\times n}$, $\overline{A} = (\overline{a}_{kj})_{n\times n}$, $\underline{B} = (\underline{b}_{kj})_{n\times n}$ and $\overline{B} = (\overline{b}_{kj})_{n\times n}$ are given known matrices in $\mathbb{C}^{n\times n}$. Therefore, the parameters $\widetilde{C}$, $\widetilde{A}$ and $\widetilde{B}$ of the complex-valued recurrent neural networks (2) are also uncertain and bounded. In the following, for represention brevity, denote $\underline{\widetilde{C}} = \mathrm{diag}\{\underline{C}, \underline{C}\}$, $\overline{\widetilde{C}} = \mathrm{diag}\{\overline{C}, \overline{C}\}$,

$$\underline{\widetilde{A}} = \begin{pmatrix} \underline{A}^R & -\overline{A}^I \\ \underline{A}^I & \underline{A}^R \end{pmatrix}, \quad \underline{\widetilde{B}} = \begin{pmatrix} \underline{B}^R & -\overline{B}^I \\ \underline{B}^I & \underline{B}^R \end{pmatrix},$$
$$\overline{\widetilde{A}} = \begin{pmatrix} \overline{A}^R & -\underline{A}^I \\ \overline{A}^I & \overline{A}^R \end{pmatrix}, \quad \overline{\widetilde{B}} = \begin{pmatrix} \overline{B}^R & -\underline{B}^I \\ \overline{B}^I & \overline{B}^R \end{pmatrix}.$$

Definition 1. The complex-valued recurrent neural networks (1) with the parameters range defined by (13) is globally robustly stable if the equilibrium point $z^* = (z_1^*, \dots, z_n^*)^T$ of the system (1) is globally asymptotically stable for all $C \in C_r$, $A \in A_r$, $B \in B_r$.

Theorem 2. Suppose that the Assumption 1 is satisfied, then the complex-valued recurrent neural network (1) subject to parameter uncertainties in (13) is globally robustly stable if there exist two diagonal matrices $P = \mathrm{diag}\{p_1, p_2, \dots, p_{2n}\} > 0$ and $D = \mathrm{diag}\{d_1, d_2, \dots, d_{2n}\} > 0$ such that

$$\begin{aligned} 2\rho - \alpha^{-1}\|P\|_2^2(\|\widetilde{A}^*\|_2 + \|\widetilde{A}_*\|_2)^2 - \|P\|_2^2(\|\widetilde{B}^*\|_2 \\ + \|\widetilde{B}_*\|_2)^2\|D^{-1}\|_2 - \beta\|H^2\|_2 > 0, \end{aligned} \tag{14}$$

where $\rho = \min_{1\le k\le n}\{p_k\underline{c}_k, p_{n+k}\underline{c}_k\}$, $\alpha = \min_{1\le k\le n}\{\underline{c}_k\}$, $\beta = \max_{1\le k\le n}\{\overline{c}_k + d_k, \overline{c}_k + d_{n+k}\}$.

Proof. Similar to Theorem 1, construct the Lyapunov functional candidate in (12) and calculate the time derivative of $V(u(t))$ along the trajectories of (3), by utilizing Lemma 1 and (9), one gets that

$$\begin{aligned} \dot{V}(u(t)) \le & -2u^T(t)P\widetilde{C}u(t) + u^T(t)P\widetilde{A}\widetilde{C}^{-1}\widetilde{A}^TPu(t) \\ & + u(t)^T(\widetilde{C} + D)H^2u(t) \\ & + u^T(t)P\widetilde{B}D^{-1}\widetilde{B}^TPu(t) \\ \le & -2\rho\|u(t)\|_2^2 + \|P\|_2^2\|\widetilde{A}\|_2^2\|\widetilde{C}^{-1}\|_2\|u(t)\|_2^2 \\ & + \|P\|_2^2\|\widetilde{B}\|_2^2\|D^{-1}\|_2\|u(t)\|_2^2 \\ & + \|\widetilde{C} + D\|_2\|H^2\|_2\|u(t)\|_2^2 \\ \le & -2\rho\|u(t)\|_2^2 + \alpha^{-1}\|P\|_2^2(\|\widetilde{A}^*\|_2 \\ & + \|\widetilde{A}_*\|_2)^2\|u(t)\|_2^2 + \|P\|_2^2(\|\widetilde{B}^*\|_2 \\ & + \|\widetilde{B}_*\|_2)^2\|D^{-1}\|_2\|u(t)\|_2^2 \\ & + \beta\|H^2\|_2\|u(t)\|_2^2. \end{aligned} \tag{15}$$

It follows from condition (14) that $\dot{V}(u(t))$ is negative definite for $u(t) \neq 0$. Hence, the uncertain complex-valued network (1) with parameters in (13) is globally robustly stable. The proof is complete. □

Remark 1. In [9], the global robust stability of complex-valued recurrent neural networks with time-delays and uncertainties has been investigated. However, the sufficient conditions obtained there only utilized the information of $\underline{C}$, while the information of $\overline{C}$ was omitted. In this paper, the given condition in Theorem 2 includes not only the information of $\underline{C}$ but also the information of $\overline{C}$.

4. Illustrative examples

In this section, an numerical example is given to demonstrate the effectiveness of the proposed criteria. Consider a two-neuron complex-valued recurrent neural network described by (1) with $\tau = 2$, $L = (-2 + i, 6 + 8i)^T$, and the parameter uncertainties are in the form of (13) with $\underline{C} = \mathrm{diag}\{2.8, 2.9\}$, $\overline{C} = \mathrm{diag}\{3.1, 3.2\}$,

$$\begin{aligned} \underline{A} &= \begin{bmatrix} -0.1 - 0.2i & 0.1 - 0.1i \\ 0.1 + 0.2i & -0.2 + 0.1i \end{bmatrix}, \\ \overline{A} &= \begin{bmatrix} 0.1 + 0.3i & 0.2 + 0.2i \\ 0.3 + 0.4i & 0.2 + 0.3i \end{bmatrix}, \\ \underline{B} &= \begin{bmatrix} -0.1 + 0.1i & -0.1 + 0.2i \\ 0.2 - 0.3i & 0.3 - 0.1i \end{bmatrix}, \\ \overline{B} &= \begin{bmatrix} 0.1 + 0.3i & 0.2 + 0.4i \\ 0.4 - 0.1i & 0.4 + 0.1i \end{bmatrix}. \end{aligned}$$

For $z_k = x_k + iy_k$ with $x_k, y_k \in \mathbb{R}$, the activation functions in (1) are taken as

$$\begin{aligned} f_k^R(z_k(t)) &= \frac{1}{2}(|x_k + 1| - |x_k - 1|), \\ f_k^I(z_k(t)) &= \frac{1}{2}(|y_k + 1| - |y_k - 1|), \ k = 1, 2. \end{aligned}$$

It is easy to verify that $f_k(\cdot)$ satisfies Assumption 1 with $r_k = 1$ and $s_k = 1$.

By resorting to Theorem 2, it is observed that there exist two diagonal matrices $P = D = I$ satisfying condition (14), i.e., $2\rho - \alpha^{-1}\|P\|_2^2(\|\widetilde{A}^*\|_2 + \|\widetilde{A}_*\|_2)^2 - \|P\|_2^2(\|\widetilde{B}^*\|_2 + \|\widetilde{B}_*\|_2)^2\|D^{-1}\|_2 - \beta\|H^2\|_2 = 0.0575 > 0$. Therefore, it follows from Theorem 2 that the complex-valued recurrent neural network (1) is globally robustly stable.

For simulation aim, we take $C = \mathrm{diag}\{2.9, 3\}$ and

$$A = \begin{bmatrix} -0.05 + 0.1i & 0.15 + 0.1i \\ 0.2 + 0.3i & -0.1 + 0.2i \end{bmatrix},$$
$$B = \begin{bmatrix} 0.05 + 0.2i & 0.1 + 0.3i \\ 0.3 - 0.2i & 0.35 + 0.05i \end{bmatrix}.$$

Figure 1 and Figure 2 illustrate the time responses of the states for the recurrent neural network (1).

5. Conclusion

In this paper, based on the Lyapunov functional method and the matrix analysis technique, not only the lower bound information but also the upper bound information of the uncertain self-feedback matrix have been considered to establish the sufficient conditions to ascertain the robust stability

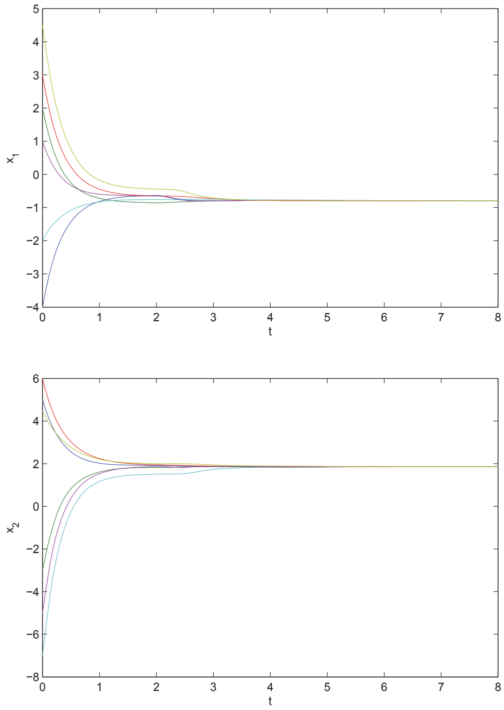

Figure 1: Trajectories of the real parts $x(t)$ of the states $z(t)$ for network (1).

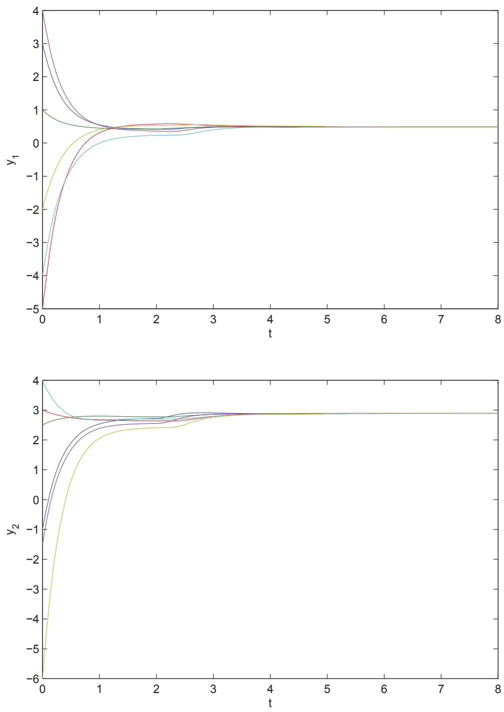

Figure 2: Trajectories of the imaginary parts $y(t)$ of the states $z(t)$ for network (1).

for the complex-valued recurrent neural networks with interval time-varying parameters and constant delay. One numerical simulation further demonstrates the effectiveness of the obtained criteria. In the near future, research topics include the stability analysis of the complex-valued recurrent neural networks with discontinuous activation functions and stochastic disturbances.

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