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Accurate and Fast Event Detection Occurrence in Planar Piecewise Affine Hybrid Systems

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Abstract—
Many switched circuits are made of linear components switched by a simple logic unit. In this paper, we use a generic planar Piecewise Affine Hybrid System (PWAHS) to model this kind of circuits. Usually, simulations are run by analytical methods (adapted to a specific simple model) or by numerical simulations that can miss some events occurrences. We propose a generic planar method to simulate PWAHSs with periodic and state dependent events. Using analytical expression, our approach can reach arbitrate accuracy in event detections without any loss. As a result, we have implemented our method in a Scilab toolbox.

1. Introduction

Switched circuits behavior is mostly simulated by pure numerical methods where precision step is increased when the system is near of a switching condition. Those numerical tools are widely used mainly because of their ease-of-use and their ability to simulate a wide range of circuits including non–linear, time–variant, and non–autonomous systems.

Even if those simulators can reach the desired relative precision for a continuous trajectory, they can miss some switching condition and then diverge drastically from the trajectory as in figure 2. This could be annoying when ones are interested by border collision bifurcations, or when local behavior is needed with a good accuracy. In those applications, an alternative is to write down analytical, or semi–analytical, trajectories and switching conditions to obtain a recurrence which is very accurate and fast to run. Building and adapting such *ad'oc* simulators represents a lot of efforts and a risk of mistakes.

Generic and accurate simulators can be proposed if we restrict to a certain class of system. A simulation tool with no loss of event is proposed in [4] and [5] for PWAHSs defined on polytopic closed sets. This class of PWA differential systems has been widely studied as a standard technique to approximate a range of non–linear system. But closed polytopic partition of the state space does not allow simulation of most switching circuits where switching frontiers are mostly single affine constraints or time–dependent periodical events.

In this paper, we focus on planar PWAHSs with such simple switching conditions which can model a family of switched planar circuits: bang–bang regulators, boost converter, charge-pump phase locked loop, ...

This class of systems has analytical trajectories that helps to build fast algorithm with no loss of events. We propose a semi-analytical solver for hybrid systems which provides :

- A pure numerical method when the system is non–nonlinear or non-planar;
- A pure analytic method when all continuous parts of the system and switching conditions can be solved symbolically. This can be the case for the boost converter [2], [9], the second order charge-pump phase locked loop [6], [8].
- A mixed method using analytical trajectories and numerical computation of the switching instant when those solutions are transcendent. This has been used for 3rd order CP-PLL [6]. It can also be the case for the buck converter [3], [9], ...

This paper is organized as follows. Section II describes the problem to be deal with and introduces a general algorithm to solve planar HSs. Section III presents the algorithm that detects events' occurrence. An illustrating example is given in Section IV. Finally, a conclusion is stated in Section V.

2. General algorithm to solve HSs

Definition of a HS (X, E, t)
A general definition of HS is presented here. This type of dynamical systems is characterized by the coexistence of two kinds of state vectors: continuous state vector $X(t)$ of real values, and discrete state vector $E(t)$ belonging to a countable discrete set $\mathcal{M}$.

Definition 1 *A continuous-time, autonomous HS is a system of the form:*

$$\begin{array}{l}\dot{X}(t) = f\left(X(t), E(t)\right), \ f : \mathcal{H} \to \mathbb{R}^n \\ E^{+}(t) = \phi\left(X(t), E(t)\right), \ \phi : \mathcal{H} \to \mathcal{M}\end{array} \quad (1)$$

$\mathcal{H} = \mathbb{R}^n \times \mathcal{M}$ is called hybrid state space. $X(t) \in \mathbb{R}^n$ is the continuous state vector of the HS at time instant t and $E(t) \in \mathcal{M} := \{1, .., M\}$ its discrete state. $E^{+}(t)$ denotes the updated discrete state right after time instant t. $\phi : \mathcal{H} \to \mathcal{M}$ describes the discrete dynamic, it is usually modellized by Petri Net. A transition from $E(t) = i$ to $E^{+}(t) = j$ is valid when the state X reaches a switching set called $\mathcal{S}_{E_i,E_j}$. Such transition are called state dependent events. A HS is called piecewise affine if for each $E \in \mathcal{M}$, $f(X, E)$ can be defined by $f(X, E) = A_E X + B_E, \forall X$.

Remark — In non autonomous HS, the function ϕ can also depend on time $\phi\left(X, E, t\right) : \mathbb{R}^n \times \mathcal{M} \times \mathbb{R} \to \mathcal{M}$. Then time dependent events can occurred and validate a transition, such as periodic events.

Hybrid system class of interest

We will consider two dimensional PWAHS ($X(t) \in \mathbb{R}^2$). f is then defined in the affine piecewise form

$$f(X(t), E(t)) = A_{E(t)} X(t) + B_{E(t)} \quad (2)$$

We will consider two kinds of events: state dependent events and periodic events.
The state dependent event transition $\mathcal{S}_{E_iE_j}$ is defined by an affine state border of the form $N'_{ij}.X < l_{ij}$. In this case an event can occur when the continuous state reaches the border of the set $\mathcal{S}_{E_iE_j} = \left\{X(t) \in \mathbb{R}^2 : N'_{ij}.X \leq l_{ij}\right\}$

Note that the state $\mathcal{S}_{E_iE_j}$ is open.

Remark — We will consider, with no loss of generality, the case where a transition occurs at time $d\mathcal{S}_{E_iE_j}$ if and only if the state $X(d\mathcal{S}_{E_iE_j})$ reaches a border of the set $\mathcal{S}_{E_iE_j}$ from the outside. Figure 1 defining a transition with the complimentary set $\mathcal{S}$ allows to detect the event in both directions. Both transitions can be met with the set $B = \mathcal{S} \cup \bar{\mathcal{S}}$. Periodic events are simply defined by time instants $t = d\mathcal{P}_{E_iE_j}$, where $d\mathcal{P}_{E_iE_j}$ belongs to the set $\mathcal{P}_{E_iE_j} = \{t : t = kT + \varphi, \ k \in \mathbb{N}\}$.
T is the period φ the phase of such periodic events.

3. Event–driven simulation of PWAHSs

The simulation will compute the hybrid state from event to event. Knowing the states $X(t_k)$ and $E(t_k^+)$, one can compute the trajectory $X(t > t_k) = \int_{t_k}^{t} f\left(X(t_k), E(t_k^+)\right) dt + X(t_k)$, assuming that the discrete state is constant $E(t > t_k^+) = E(t_k^+)$.

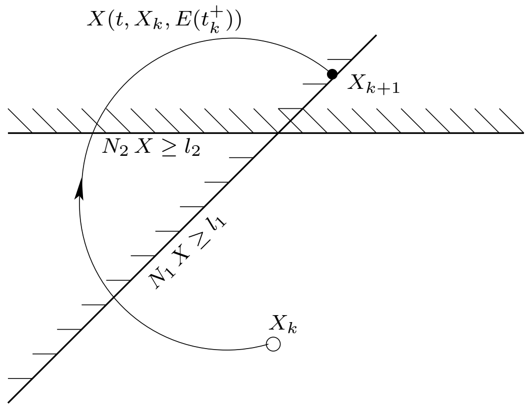

Figure 1: Oriented polytopic state dependent transitions.

Then the following algorithm runs the simulation determining the dates at the next event as the smallest:

```
Data: t_k, X_k, E_k.
while t < t_fin do
    Compute all events' dates dS_{E_iE_j} and dP_{E_iE_j}
    ;
    t_{k+1} = min(dS_{E_iE_j}, dP_{E_iE_j});
    X_{k+1} = f (X_k, E_k, t_{k+1});
    E_{k+1} = φ (X_k, E_k, t_{k+1});
end
```

Algorithm 1: Algorithm computing hybrid state at t_{k+1}

4. Algorithm to detect event occurrence

We consider an affine Cauchy problem in $\mathbb{R}^2$:

$$\left\{\begin{array}{ll}\dot{X}(t) = AX(t) + B, & t > t_0 \\ X(t_0) = X_0 & \end{array}\right. \quad (3)$$

where X_0 is the initial value. We compute the smallest strictly positive time t^* so that the trajectory of $X(t)$ intersects the fixed border B_i arriving from the part of the plan where $N'_i.X < l_i$.

The function $f_i(t) = N'_i.X(t) - l_i$ defines the condition guard for a border B_i. Thus, the problem can be formulated as follows:
Find the smallest $t^* > 0$ such that

$$\left\{\begin{array}{ll}\exists\, t = t^*, & f(t^*) = 0 \\ \exists\, \delta > 0, \forall t \in]t^* - \delta, t^*[\,, & f(t) < 0\end{array}\right. \quad (4)$$

If f_i does not have any strictly positive root or the last condition is not satisfied, t_i^* is given the infinite value.

4.1. Analytical trajectories

The analytical trajectory $X(t)$ is given by the general integral form:

$$X(t) = \mathrm{e}^{A(t-t_0)} X_0 + \int_{t_0}^{t} \mathrm{e}^{A(t-s)} B \mathrm{d}s \quad (5)$$

When A is invertible, the expression becomes linear:

$$X(t) + A^{-1}B = \mathrm{e}^{A(t-t_0)}(X_0 + A^{-1}B) \quad (6)$$

The analytical expression of exponential matrix e^{At} takes two forms depending on whether the eigenvalues p_1 and p_2 of the matrix A are equals or not:
If $p_1 \neq p_2$, then

$$\mathrm{e}^{At} = \frac{(p_1\mathbb{I} - A^\circ)}{p_1 - p_2}\ \mathrm{e}^{p_1 t} - \frac{(p_2\mathbb{I} - A^\circ)}{p_1 - p_2}\ \mathrm{e}^{p_2 t} \quad (7)$$

If $p_1 = p_2 = p$, then

$$\mathrm{e}^{At} = (\mathbb{I} + (p - A^\circ)\ t)\ \mathrm{e}^{pt} \quad (8)$$

where: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $A^\circ = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

Using these expressions, we can determine the function $f(t)$ of the problem (4) to be in the form:

$$f(t) = a_1 + a_2\,t + a_3\,t^2 + (a_4 + a_5\,t)\,\mathrm{e}^{p_1\,t} + a_6\,\mathrm{e}^{p_2\,t}$$

where a_i are real scalars.

Depending on eigenvalues p_1 and p_2, there is five cases that determines the values of coefficients a_i as shown in Table 1.

$f(t) = a_1 + \ldots$	$p_1 \in \mathbb{R}^*$	$p_1 = 0$
$p_2 \in \mathbb{R}^*$	$a_4\,\mathrm{e}^{p_1 t} + a_6\,\mathrm{e}^{p_2 t}$	$a_2\,t + a_6\,\mathrm{e}^{p_2 t}$
$p_2 = 0$	$a_2\,t + a_4\,\mathrm{e}^{p_1 t}$	$a_2\,t + a_3\,t^2$
$p_1 = \overline{p_2} \in \mathbb{C}^*$	$a_4\mathrm{e}^{p_1 t} + a_5\mathrm{e}^{p_2 t}$, with $a_5 = \overline{a_4} \in \mathbb{C}^*$	
$p_1 = p_2 \in \mathbb{R}^*$	$(a_4 + a_5\,t)\mathrm{e}^{p_1 t}$	

Table 1: Expressions of $f(t)$ depending on eigenvalues p_1 and p_2.

Remark — Coefficients a_i are real scalars that depend on the eigenvalues p_1 and p_2, the initial point X_k and the border parameters N_i and l_i.

In some cases, ($p_1 = p_2 = 0$, grey cell in Table 1) roots of $f(t)$ can be found analytically and the problem is solved with machine precision.

In other cases,solution can not be found with classical functions and then a numeric algorithm should be used. Using classical methods like newton does not guaranty existence or convergence of the smallest positive root. To meet these conditions, let use analytical roots of the derivative function $f'(t)$ expressed in Table 2.

We can then compute analytically the set L of ordered roots of $f'(t)$, those roots determines monotone intervals of $f(t)$ The following algorithm is used to return the solution t^* when it exists or the value ∞ if not.

$f'(t) = \ldots$	$p_1 \in \mathbb{R}^*$	$p_1 = 0$
$p_2 \in \mathbb{R}^*$	$a_4\,p_1\,\mathrm{e}^{p_1 t} + a_6\,p_2\,\mathrm{e}^{p_2 t}$	$a_2 + a_6\,p_2\,\mathrm{e}^{p_2 t}$
$p_2 = 0$	$a_2 + a_4\,p_1\,\mathrm{e}^{p_1 t}$	$a_2 + 2\,a_3\,t$
$p_1 = \overline{p_2} \in \mathbb{C}^*$	$a_4\,p_1\,\mathrm{e}^{p_1 t} + \overline{a_4 p_1}\mathrm{e}^{\overline{p_1} t}$, with $a_4 \in \mathbb{C}^*$	
$p_1 = p_2 \in \mathbb{R}^*$	$(a_4\,p_1 + a_5 + a_5\,p_1\,t)\ \mathrm{e}^{p_1 t}$	

Table 2: Expressions of $f'(t)$ depending on eigenvalues p_1 and p_2: all roots are analytical

```
Data: N_i, l_i, A, B, X_k
Result: construct the set L, compute t*
T ← {0, L, ∞};
t* ← ∞;
for i ← 1 to (card(T) − 1) do
    if f(T(i)) < 0 & f(T(i + 1)) > 0 then
        t* ← solve [T(i), T(i + 1)];
        Break;
    end
end
```

Algorithm 2: Algorithm computing t^* when a solution is transcendent

Remark — When $(p_1, p_2) \in \mathbb{C}^* \times \mathbb{C}^*$ the set L is infinite: when the real part of p_i is positive, the algorithm will end by finding a root. In the other case the set L should be reduced to its 3 first elements, to find a crossing point when it exists.

5. Example

The above semi–analytical algorithm was implemented and tested on the following simple PWAHS (fig.3):

$$\begin{aligned} S_1:\quad \dot{X}(t) &= \begin{bmatrix} 0.1 & 1 \\ -1 & 0.1 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ S_2:\quad \dot{X}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \end{aligned} \quad (9)$$

with state events:

$$\begin{aligned} \mathcal{S}_{E_1E_2} &= \left\{X \in \mathbb{R}^2 : [1\ 0]\,X < -0.513036\right\} \\ \mathcal{S}_{E_2E_1} &= \left\{X \in \mathbb{R}^2 : [0\ 1]\,X < 1\right\} \end{aligned} \quad (10)$$

In order to illustrate the effectiveness of our approach, we have compared our simulation results with those obtained using the numerical solver Scicos (ODE “lsoda”). Figure 2 that depicts the two different simulations with initial condition $X_0 = [0; 0.17]$ and the same relative and absolute precision, shows that Scicos trajectory represented by a dashed curve in the plot, has skipped a state event that has been detected by our algorithm at $t = 11.096993$s.

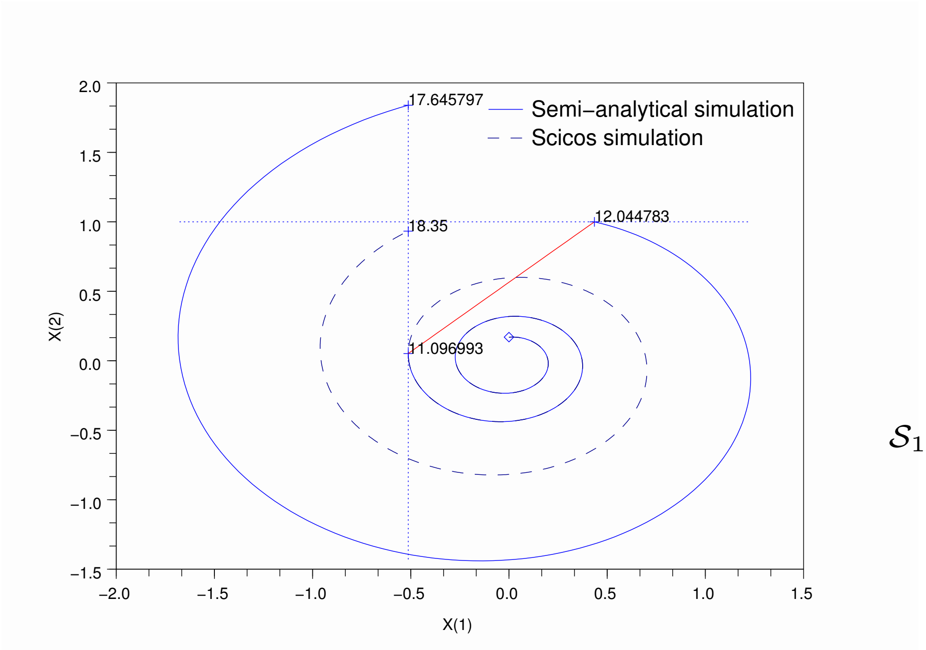

Figure 2: Scicos simulation versus semi–analytical simulation

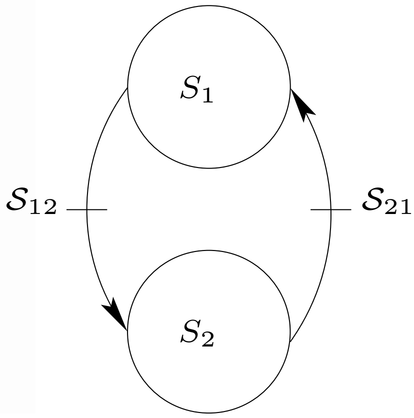

Figure 3: Hybrid automaton example.

6. Conclusion

In this paper, we have demonstrated an accurate and fast method to determine events occurrence for planar piecewise affine hybrid systems. As a result, we have implemented our algorithm in a scilab toolbox (free download on authors web pages) that will be extended in the near future to add analysis tools such as displaying the bifurcation and parametric diagrams. This algorithm takes advantage of analytical form that appear in the planar case. This can not be extended to higher dimension, in this case algorithm presented in [4] should be adapted in a bounded space with special cares concerning the event direction.

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