2015 International Symposium on Nonlinear Theory and its Applications
NOLTA2015, Kowloon, Hong Kong, China, December 1-4, 2015

Controllability of Complex Dynamical Networks with Multidimensional Node Dynamics

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Abstract– In a complex dynamical network, one node of the network often has multidimensional dynamics. In this paper, the controllability of complex dynamical networks with multidimensional node dynamics is studied. Using the Maximum Matching principle, the minimum number of controlled nodes is obtained. Rigorous criteria are deduced to find out the driver nodes and some simulations are given to effectively verify the proposed schemes.

1. Introduction

A complex dynamical network is composed of numerous nodes, which are driven by time-dependent characteristics diversely [1-3]. Multiple existences of complex dynamical networks have extended everywhere from social networks [4], communication networks [5], power grids [6], the Internet [7], etc, which have a far-reaching influence on our daily life. For a long time, people have been arguing that how to control a complex dynamical network effectively and have developed many methods, including adaptive feedback control [8], pinning control [9], impulse control [10], event-triggered control [11], etc.

Recently, much attention has been paid on the controllability of complex dynamical networks [12-17]. Liu *et al*. [12] firstly studied this problem, and proposed the so-called minimum input theorem. It is used to decide the minimum number of driver nodes by using the Maximum Matching principle in graph theory. Their most contribution is that they introduce the Maximum Matching principle into the study of network controllability. In [12], the dynamic of each node in the network is described by the one-dimensional linear system, thus the network they considered is actually a multidimensional linear system. However, in most physical complex dynamical networks, the node often has multidimensional dynamics. In [13], the control centrality is introduced to quantify the ability that one single node fully control a directed and weighted complex network. An exact controllability example of complex dynamical networks is introduced by Yuan *et al*. [15]. Based on the eigenvalues and rank of the network matrix, they shown that the minimum number of driver nodes is fixed and determined by the maximum geometric multiplicity of the network matrix. Their framework is valid for both directed and undirected networks. A perturbation approach to optimizing the controllability of complex dynamical networks is proposed in [16]. They selected proper locations judiciously at which as few as possible links are added so that the full rank condition could be satisfied, that is, the whole perturbed complex network could be fully controlled by only one external input. However, most existing researches on the controllability of complex dynamical networks [12-16] are still concentrated on the network characterized by one-dimensional node dynamics, which is impractical to most complex networks whose node dynamics are often multidimensional. The node dynamics should be the vital factor to determine the controllability of complex networks [17]. If all nodes in a complex dynamical network could perceive and use their own states, which means each node has its own self-loop, then the whole network is perfect matching and its controllability is not determined by the degree distribution but the node dynamics.

In this paper, we investigate the controllability problem of the complex dynamical network assuming that the node of the network has n-dimensional $(n \geq 1)$ dynamics. By using the Maximum Matching principle, we achieve the minimum number of driver nodes. Corresponding criteria have been obtained. The rest parts of this paper are arranged as follows. In Section 2, considering the states of driver nodes could be fully controlled, the controllability of complex networks with n-dimensional node dynamics is studied. In Section 3, several simulation examples are given to demonstrate the validity of the proposed schemes. Some conclusions are provided in Section 4.

2. Controllability of general complex networks

Consider a complex dynamical network $G(D)$ whose topology structure is represented by D consisting of N nodes with n-dimensional dynamics, shown in Eq. (1):

$$\dot{x}_i(t) = f_i(x_i(t)) + \sum_{j=1}^{N} d_{ij}\Gamma_{ij}x_j(t), \tag{1}$$

where $i, j = 1,2,...,N$, $x_i(t) = \left(x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t)\right)^T \in R^n$ is the state vector of the i^{th} node, $f_i : R \times R^n \to R^n$ is a smooth nonlinear vector field, which describes the dynamic behavior of the i^{th} node. $D = \left(d_{ij}\right)_{N\times N} \in R^{N\times N}$ is the configuration matrix which implies the interactive structure underlying the complex network. $d_{ij} \neq 0$ if there exists a directed connection from node i to node j ($i \neq j$), otherwise $d_{ij} = 0$. Define the diagonal elements of D as $d_{ii} = -\sum_{j=1, j\neq i}^{N} d_{ij}$. $\Gamma_{ij} = \left(\tau_{pq}\right)_{ij} \in R^{n\times n}$ is the inner coupling matrix whose elements represent relations between each node and the whole network, suggesting that how combinations of state variables in different nodes influence the complex network.

For generality, three hypotheses are introduced here:

Hypothesis 1 (H1) The topology matrix D is symmetric, i.e., $d_{ij} = d_{ji}$, which means all the interconnections between any two nodes are undirected.

Hypothesis 2 (H2) The inner coupling matrix Γ_{ij} is simplified as identity matrix which describes a mostly common type of coupling. The coupling manner between any two nodes is actually reflected in the interactions between corresponding states in each node.

For simplicity, we assume the node dynamics of the network as an n-dimensional linear system. The complex network $G(D)$ could be expressed by Eq. (2) from Eq. (1):

$$\dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{N} d_{ij}\Gamma x_j(t) + b_i u_i(t), \tag{2}$$

where $u_i(t) = \left(u_{i1}(t), u_{i2}(t), \cdots, u_{im}(t)\right)^T \in R^m$ is called the input vector, which is injected to control the corresponding node through the input matrix $b_i \in R^{n\times m}$. $A = \left(a_{ij}\right)_{n\times n} \in R^{n\times n}$ is the system matrix. It is assumed that there exist zero elements in A, which means the linear system is not fully coupling.

Hypothesis 3 (H3) For a given complex network, if there exist certain nodes needed to be controlled, that is, every state of those nodes then would be imposed control inputs on independently.

In the following, the Maximum Matching principle is utilized to find out which nodes and state variables are unmatched such that we could impose effective inputs accordingly. For the complex network $G(D)$ composed of N n-dimensional dynamical nodes, it could be view as n simple artifact networks $\{G_k(D) \mid k = 1,2,...,n\}$, as Fig. 1 shows:

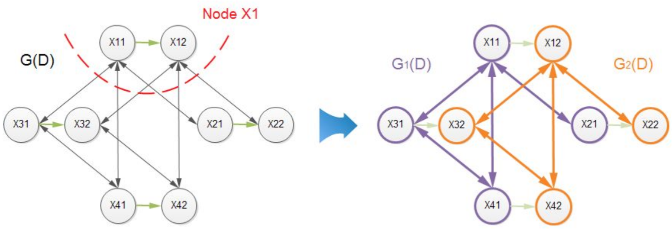

Fig. 1 $G(D)$ is viewed as $\{G_k(D) \mid k = 1,2\}$.

Select one $G_k(D)$ arbitrarily and implement the Maximum Matching principle on it, it is easy to decide whether $G_k(D)$ is perfect matching: if so, any one node could be chosen as the driver one; if not, unmatched nodes would be imposed control inputs on directly. If only those nodes singled out through the Maximum Matching principle are injected effective inputs accordingly, the origin complex network $G(D)$ would be controllable.

Theorem 1 Suppose that (H1), (H2) and (H3) hold. For the complex network $G(D)$ with n-dimensional node dynamics, its controllability is equivalent to the counterpart of $G_k(D)$ with one-dimensional node dynamics.

Proof The key to address the structural controllability problem of the complex network with n-dimensional node dynamics lies in 'isolating' the origin network $G(D)$ into n artifact subnetworks $\{G_k(D) \mid k = 1,2,...,n\}$. There are two prerequisites for the so-called 'isolation': (1) (H2) holds. Whether the node states could be matched or not completely depends on the coupling structure D rather than the node dynamics A (2) (H3) holds. (H3) guarantees that all $\{G_k(D)\}$ would not differ from each other after imposing control inputs on certain nodes. In every $G_k(D)$, it takes the same way to fulfill the maximum matching and eliminate the unmatched nodes with external inputs. Once these two prerequisites are satisfied, $G(D)$, with n-dimensional node dynamics, could be viewed as n independent subnetworks $\{G_k(D)\}$, with one-dimensional node dynamics. By now one arbitrary $G_k(D)$ would be available for the Maximum Matching principle. If one $G_k(D)$ is controllable, the others would be either. Then the origin complex network $G(D)$ would be controllable. The proof is done. □

Theorem 1 has essential interpretations to the controllability of undirected complex networks attached multidimensional node dynamics, which is significant in practical applications.

3. Simulation examples

A complex network $G(D)$ composed of ten dynamical nodes is chosen here for simulation, which is shown as Eq. (3):

$$\dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{10} d_{ij}\Gamma x_j(t), \quad (3)$$

The star-shaped structure is selected as the node dynamics, which is described by the system matrix A as shown in Fig. 2:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad x_i(t) = \begin{bmatrix} x_{i1}(t) \\ x_{i2}(t) \\ x_{i3}(t) \\ x_{i4}(t) \end{bmatrix},$$

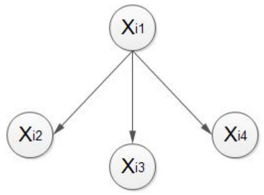

Fig. 2 The node dynamics.

The topology structure of $G(D)$ is set as D, which is shown in Fig. 3:

$$D = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

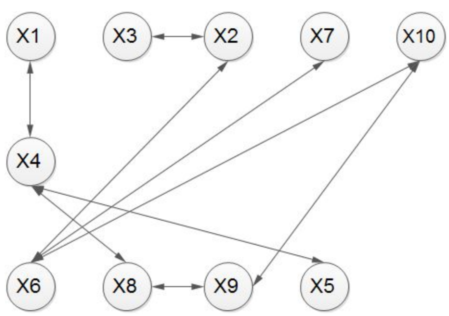

Fig. 3 The topology structure of $G(D)$.

It evidently satisfies Hypothesis 1 and 2. If every state of the driver node is controlled directly, namely Hypothesis 3 holds, then $G(D)$ could be viewed as four subnetworks $\{G_1(D), G_2(D), G_3(D), G_4(D)\}$, which have the same coupling structure D as shown in Fig. 4:

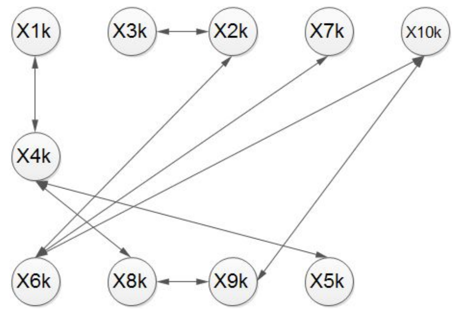

Fig. 4 The coupling structure of $\{G_k(D) \mid k = 1,2,3,4\}$.

Apply the Maximum Matching principle on an arbitrary $G_k(D)$ and its bipartite graph is demonstrated as Fig. 5:

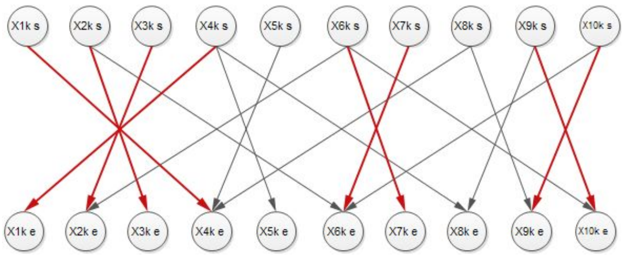

Fig. 5 The bipartite graph of $G_k(D)$.

From Fig. 5, it seems obviously that nodes $\{X5, X8\}$ are unmatched, which should be chosen as driver nodes:

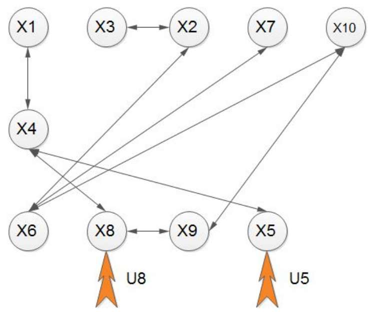

Fig. 6 The controlled network $G(D)$.

Eq. (3) is rewritten as Eq. (4), with injected inputs $\{U5, U8\}$, which is shown as follows:

$$\dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{10} d_{ij}\Gamma x_j(t) + b_i u_i(t), \quad (4)$$

where $u_i(t) = \begin{bmatrix} u_{i1}(t) \\ u_{i2}(t) \\ u_{i3}(t) \\ u_{i4}(t) \end{bmatrix}$, $b_5 = b_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $b_i = O$, $i = 1,2,3,4,6,7,9,10$.

Fig. 7 illustrates that every state of all nodes in the complex network $G(D)$ converges eventually to zero after imposing proper inputs on fully controlled driver nodes, which verifies the correctness of the proposed criteria that only controlling the minimum number of nodes could control the whole network with multidimensional node dynamics.

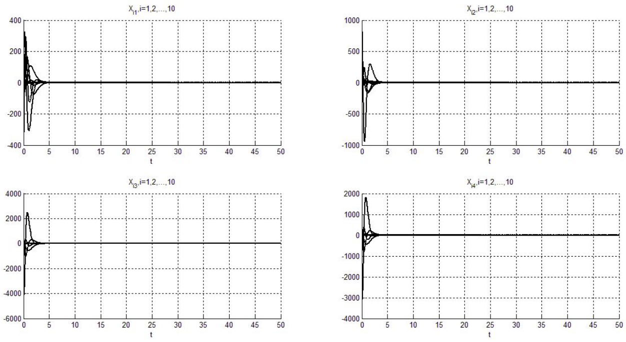

Fig. 7 Node states of the controlled network $G(D)$ with fully controlled driver nodes.

5. Conclusions

The structural controllability of general complex networks with multidimensional node dynamics is investigated specifically in this paper. The Maximum Matching principle plays an important role in picking out the minimum number of nodes which should be controlled directly. Rigorous criteria are deduced and proofs are given accordingly. Simulation examples verify the feasibility of the control scheme we proposed.

On the one hand, this paper provides a fully sufficient condition; on the other hand, it might be still too ideal to be met in practice. Thus, the more subtle problem that whether it is possible to control complex dynamical networks with only partial states of driver nodes would be investigated in our future work.

Acknowledgments

The authors would like to express sincere appreciation to NOLTA2015 organizing committee members for their fruitful suggestions and comments.

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