2015 International Symposium on Nonlinear Theory and its Applications
NOLTA2015, Kowloon, Hong Kong, China, December 1-4, 2015

Stabilization of a novel fractional-order chaotic financial system via state feedback technique

Chengdai Huang† and Jinde Cao†,‡

†Research Center for Complex Systems and Network Sciences, and Department of Mathematics, Southeast University, Nanjing 210996, China
‡Research Center for Complex Systems and Network Sciences, and Department of Mathematics, Southeast University, Nanjing 210996, China,
Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
Email: huangchengdai@163.com, jdcao@seu.edu.cn

Abstract—The stabilization of a novel fractional-order chaotic financial systems involving market confidence is concerned in this paper. Based on fractional stability theory, we apply a state feedback controller to stabilize such system to desirable equilibrium and the equilibriums of the original system are kept completely. Finally, numerical examples are given to demonstrate the correctness of the theoretical results.
Keywords: Chaotic system, Fractional-order financial system, State feedback, Stability, Stabilization.

1. Introduction

Researches have shown that fractional derivatives provide an excellent tool for describing the memory and hereditary properties of various materials and processes. Recently, control of fractional-order financial systems have fascinated many researchers [1, 2, 3]. In [1], the sliding control law was achieved to stabilize the fractional-order system by the sliding control strategy. In [3], it is shown that an appropriate time delay can enhance or suppress the emergence of chaotic or periodic motions.

It is well known that the most important factor to influencing or rescue the economic crisis is confidence, the confidence of the people was very important [4, 5]. The author revealed that confidence is easy to lose and hard to gain in [4]. It is shown that the economic emerges, the people losing confidence would bring a series of chain reaction and vicious cycle in [6], such as stopping consuming, canceling investment, and reducing producing, which will lead to less confidence, less investment, less employment, more layoffs, more stock of product, more deficit. Governments always do their best to provide a balancing platform that fosters savings and investments by adopt some strategies. It is important to reduce or eliminate the chaos phenomenon in fractional financial systems via stabilization to improve the performance of economy, such as preserving stability. Hence, it is meaningful to investigate the stabilization of fractional financial systems.

To our knowledge, the stabilizations of fractional-order financial systems via state feedback are not fully studied. Motivated by this fact, in this paper, we present a state feedback controller to stabilize chaos of the financial system to the desirable equilibriums. The study indicates the proposed method can effectively eliminate chaos and stabilize the financial market.

2. Preliminaries

In this section, we present the widely accepted Caputo definition of fractional derivative. The main theoretical tools for the qualitative analysis of fractional dynamical systems are given in [8][9].

Definition 1.[7] The Caputo fractional derivative of order α of a continuous function $f(t)$ with respect to t is defined as follows

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}, \qquad (1)$$

where m is the first integer larger than α, $m-1 \le \alpha \le m$, $\Gamma(\cdot)$ is the gamma function.

Definition 2.[9] Consider the equilibrium points of fractional system

$$D^\alpha x_i(t) = f_i(x_i(t)), \quad i = 1, 2, \dots, n, \qquad (2)$$

where $x_i(t) = (x_1(t), x_2(t), \dots, x_n(t))$, $f_i(t) = (f_1(t), f_2(t), \dots, f_n(t))$.
The equilibrium solutions are defined by $f_i(x_i^*) = 0$, therefore, we can get the equilibrium points $(x_1^*, x_2^*, \dots, x_n^*)$.

3. Model description

In [6], the authors modelled a novel financial system with market confidence, this model is described by

$$\begin{cases} D^{\alpha}x_1 = x_3 + (x_2 - a)x_1 + m_1x_4, \\ D^{\alpha}x_2 = 1 - bx_2 - x_1^2 + m_2x_4, \\ D^{\alpha}x_3 = -x_1 - cx_3 + m_3x_4, \\ D^{\alpha}x_4 = -x_1x_2x_3, \end{cases} \tag{3}$$

where $0 < \alpha \leq 1$, x_1 stands for the interest rate, x_2 represents the investment demand, x_3 denotes the price index, x_4 denotes the market confidence, a is the saving amount, b is the cost per investment, c is the demand elasticity of commercial market, a, b and c are nonnegative. m_1, m_2 m_3 are the impact factors, the detailed account to the system (3) can be found in [6]. When $\alpha = 0.95$, $a = 2.1$, $b = 0.01$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$, system (3) indicates chaotic attractor in [6], which is depicted in Figure.1.

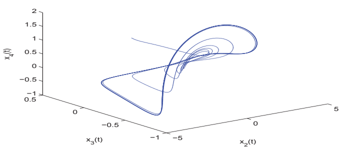

Figure 1: The chaotic attractor of system (3).

Following the ideas developed in [10], in this paper, we will design a linear state feedback controller to stabilize the fractional-order system (3). The controlled fractional-order chaotic system is given by

$$\begin{cases} D^{\alpha}x_1 = x_3 + (x_2 - a)x_1 + m_1x_4 - k_1(x_1 - x_1^*), \\ D^{\alpha}x_2 = 1 - bx_2 - x_1^2 + m_2x_4 - k_2(x_2 - x_2^*), \\ D^{\alpha}x_3 = -x_1 - cx_3 + m_3x_4 - k_3(x_3 - x_3^*), \\ D^{\alpha}x_4 = -x_1x_2x_3 - k_4(x_4 - x_4^*), \end{cases} \tag{4}$$

where k_1, k_2 k_3, k_4 are feedback gain, $(x_1^*, x_2^*, x_3^*, x_4^*)$ is a desirable equilibrium.

4. Main results

In this section, we will discuss the equilibriums of the uncontrolled system (3) and consider the stabilization of the controlled system (4). Obviously, the controlled system (4) and the uncontrolled system (3) have identical equilibriums.

To discuss conveniently the equilibrium of system (3), the assumptions are given as follows.

(A_1) $b > 0$.
(A_2) $cm_1 + m_3 = 0$.
(A_3) $cm_1 + m_3 \neq 0$.
(A_4) $m_1 = 0$, $m_3 = 0$.
(A_5) $m_1 \neq 0$, $m_3 = 0$.
(A_6) $m_3 \neq 0$ and $m_2^2 - 4m_3[m_3(ab-1) - m_1b] \geq 0$.

In the following, we discuss the equilibriums of uncontrolled system (3). Considering the steady state of uncontrolled system (3), we have

$$\begin{cases} x_3 + (x_2 - a)x_1 + m_1x_4 = 0, \\ 1 - bx_2 - x_1^2 + m_2x_4 = 0, \\ -x_1 - cx_3 + m_3x_4 = 0, \\ -x_1x_2x_3 = 0. \end{cases} \tag{5}$$

To obtain the equilibrium of the system (3), it is necessary to solve Eq.(5). From the last equation of Eq.(5), it can be seen that several cases should be considered separately.

Assume that $x_1 = 0$, then Eq.(5) becomes

$$\begin{cases} x_3 + m_1x_4 = 0, \\ 1 - bx_2 + m_2x_4 = 0, \\ -cx_3 + m_3x_4 = 0. \end{cases} \tag{6}$$

According to Eq.(6), we easily derive that
(i) If $b > 0$ and $cm_1 + m_3 \neq 0$, then $x_1 = 0$, $x_2 = \frac{1}{b}$, $x_3 = 0$, $x_4 = 0$, then the system (3) has an equilibrium $(0, \frac{1}{b}, 0, 0)$.
(ii) If $b > 0$ and $cm_1 + m_3 = 0$, then $x_1 = 0$, $x_2 = \frac{1+m_2\gamma_1}{b}$, $x_3 = -m_1\gamma_1$, $x_4 = \gamma_1$. Therefore, the system (3) has an equilibrium $(0, \frac{1+m_2\gamma_1}{b}, -m_1\gamma_1, \gamma_1)$.

Similar to the above discussion, we can easily discuss others cases: $x_2 = 0$, $x_3 = 0$, $x_1 = x_2 = 0$, $x_1 = x_3 = 0$, $x_3 = x_3 = 0$, $x_1 = x_2 = x_3 = 0$. Due to page limit, the discussions are omitted.

From the above analysis, we can conclude the following results:

Theorem 1. Consider system (3), the following results hold.
1) If (A_1) and (A_2) hold, then system (3) has an equilibrium $(0, \frac{1+m_2\gamma_1}{b}, -m_1\gamma_1, \gamma_1)$.
2) If (A_1) and (A_3) hold, then system (3) has an equilibrium $(0, \frac{1}{b}, 0, 0)$.
3) If (A_2) holds, then system (3) has an equilibrium $(0, 0, \frac{m_1}{m_2}, -\frac{1}{m_2})$.
4) If (A_3) holds, then system (3) has an equilibrium $(\frac{(cm_1+m_3)\gamma_2}{1+ac}, 0, \frac{(am_3-m_1)\gamma_2}{1+ac}, \gamma_2)$, where $\gamma_2 = \frac{m_2(1+ac)^2 \pm \sqrt{m_2^2(1+ac)^4 + 4[(cm_1+m_3)(1+ac)]^2}}{2(cm_1+m_3)^2}$.
5) If (A_1) and (A_4) hold, then system (3) has an equilibrium $(0, \frac{1+m_2\gamma_3}{b}, 0, \gamma_3)$.
6) If (A_1) and (A_5) hold, then system (3) has an equilibrium $(0, \frac{1}{b}, 0, 0)$.
7) If (A_1) and (A_6) hold, then system (3) has an equilibrium $(m_3\gamma_4, \frac{1-m_3^3\gamma_4^2+m_2\gamma_4}{b}, 0, \gamma_4)$,

where $\gamma_4 = \frac{m_2 \pm \sqrt{m_2^2 - 4m_3[m_3(ab-1) - m_1 b]}}{2m_3^2}$.

Remark 1: In Theorem 1, in general, the number of the equilibrium of the system (6) is different for given parameter values a, b, c, m_1, m_2 and m_3.

It is not difficult to see that characteristic equation of the controlled system (4) at the equilibrium point $(x_1^*, x_2^*, x_3^*, x_4^*)$ is

$$\lambda^4 + P_1\lambda^3 + P_2\lambda^2 + P_3\lambda + P_4 = 0, \qquad (7)$$

where

$$\begin{aligned}
P_1 =&(k_1 + k_2 + k_3 + k_4) + a + b + c - x_2^*,\\
P_2 =&(k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4) + [a(k_2\\
&+ k_3 + k_4) + b(k_1 + k_3 + k_4) + c(k_1 + k_2 + k_4)]\\
&- (k_2 + k_3 + k_4 + b + c)x_2^* + m_1x_2^*x_3^* + m_2x_1^*x_3^*\\
&+ m_3x_1^*x_2^* + 2(x_1^*)^2 + ab + bc + ac + 1,\\
P_3 =&(k_1k_2k_3 + k_1k_2k_4 + k_1k_3k_4 + k_2k_3k_4) + a(k_2k_3\\
&+ k_3k_4 + k_2k_4) + b(k_1k_3 + k_1k_4 + k_3k_4) + c(k_1k_2\\
&+ k_1k_4 + k_2k_4) + ab(k_3 + k_4) + bc(k_1 + k_4) + ac\\
&\cdot (k_2 + k_4) + k_2 + k_4 + (bm_3 + m_3k_2 + am_3 + am_2\\
&+ m_3k_1 - m_1)x_1^*x_2^* + (cm_2 + m_2k_3 + m_2k_1)x_1^*x_3^*\\
&+ (cm_1 + m_1k_3 + bm_1 + m_1k_2 + m_3)x_2^*x_3^* - (k_2k_3\\
&+ k_2k_4 + k_3k_4 + bk_4 + ck_4 + bk_3 + ck_2 + bc)x_2^*\\
&+ 2(k_3 + k_4 + c)(x_1^*)^2 - 2m_1(x_1^*)^2 - m_3x_1^*(x_2^*)^2 + b,\\
P_4 =&k_1k_2k_3k_4 + ak_2k_3k_4 + bk_1k_3k_4 + ck_1k_2k_4 + abk_3k_4\\
&+ bck_1k_4 + ack_2k_4 + k_2k_4 + (abc + b)k_4 + (-bm_1\\
&- m_1k_2 + abm_3 + am_3k_2 + bm_3k_1 + m_3k_1k_2)x_1^*x_2^*\\
&+ (bm_3 + m_3k_2 + bcm_1 + bm_1k_3 + cm_1k_2 + m_1m_2\\
&\cdot m_3)x_2^*x_3^* + m_2(1 + ac + ak_3 + ck_1 + k_1k_3)x_1^*x_3^*\\
&- m_3(b + k_2)x_1^*(x_2^*)^2 - 2(cm_1 + m_3)(x_1^*)^2x_3^*\\
&- m_2(x_1^*)^2x_2^* - 2(m_1k_3 - ck_4 - k_3k_4)(x_1^*)^2\\
&+ 2m_3(x_1^*)^3x_2^* - k_4(bc + bk_3 + ck_2 + k_2k_3)x_2^*.
\end{aligned}$$

According to Theorem 1, we can obtain the following Theorem 2 without difficulty.

Theorem 2. If $a = 2.1$, $b = 0.01$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$, then the controlled system has the following five equilibriums:

$$\begin{aligned}
E_1^* &= (0, 100, 0, 0),\\
E_2^* &= (2.1788, 0, -0.3426, 0.5855),\\
E_3^* &= (-0.4589, 0, 0.0727, -0.1233),\\
E_4^* &= (3.2245, -1.7182, 0, 1.4657),\\
E_5^* &= (-0.3155, -1.7182, 0, -0.1434).
\end{aligned}$$

Applying the results in [10] and together with Theorem 2 and (7), we can obtain the following Theorem.

Theorem 3. When $a = 2.1$, $b = 0.01$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$, $E_1^* = (0, 100, 0, 0)$, the controlled system (4) converges to the equilibrium E_1^* if the feedback gain k_1, k_2, k_3, k_4 satisfy following conditions:

$$P_1 > 0, P_1P_2 - P_3 > 0, P_1P_2P_3 - P_3^2 - P_1^2P_4 > 0, P_4 > 0.$$

Proof. Substituting $a = 2.1$, $b = 0.01$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$ $x_1^* = 0$, $x_2^* = 100$ and $x_3^* = 0$ into Eq.(7), then we can obtain the coefficients of Eq.(7). According to Routh-Hurwitz criterion of fractional-order differential equation, from Eq.(7), we obtain that

$$P_1 > 0, \quad \begin{vmatrix} P_1 & 1 \\ P_3 & P_2 \end{vmatrix} > 0, \begin{vmatrix} P_1 & 1 & 0 \\ P_3 & P_2 & P_1 \\ 0 & P_4 & P_3 \end{vmatrix} > 0, P_4 > 0.$$

Hence, $P_1 > 0, P_1P_2 - P_3 > 0, P_1P_2P_3 - P_3^2 - P_1^2P_4 > 0, P_4 > 0$. The proof of Theorem 3 is completed. □

Similarly, if the proper feedback gains k_1, k_2, k_3, k_4 are given, the controlled system (4) converges to the desirable equilibriums E_2^*, E_3^*, E_4^*, E_5^*, respectively, which are omitted here.

Remark 2. The obtained conditions of Theorem 3 are merely sufficient conditions.

Remark 3. Comparing with [1, 6], the controllers used in our paper are more convenient and flexible, for each case, a variety of options can be adopted to stabilize the controlled system (4) to the desirable equilibrium.

5. Illustrative examples

To evaluate the effectiveness of proposed scheme, simulation results are presented based on the fractional predictor algorithm developed in [11].

The selected parameters all are taken from [6], $\alpha = 0.95$, $a = 2.1$, $b = 0.01$, $c = 2.6$, $m_1 = 8.4$, $m_2 = 6.4$, $m_3 = 2.2$. [6] displays that the uncontrolled system (4) is chaotic. Choosing the feedback gains $(k_1, k_2, k_3, k_4) = (100, 6, 2, 4)$, which satisfies Theorem 3, from Eq.(7), we get $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-2.6, -4, -4.1, -6.01)$. Figure.2 displays that the equilibrium $E_1^* = (0, 100, 0, 0)$ of the controlled system (4) is asymptotically stable.

In addition, we may select other appropriate gain parameters k_1, k_2, k_3, k_4, the controlled system (4) converges to the desirable equilibriums E_2^*, E_3^*, E_4^*, E_5^*, respectively, which are demonstrated in Figure.3-6.

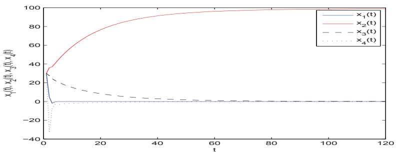

Figure 2: The trajectories of the controlled system (4) converge to the equilibrium $E_1^* = (0, 100, 0, 0)$ when $\alpha = 0.95$, where $k_1 = 100$, $k_2 = 6$, $k_3 = 2$, $k_4 = 4$.

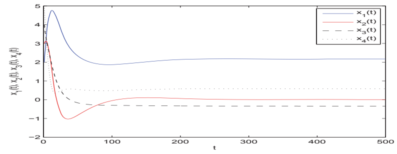

Figure 3: The trajectories of the controlled system (4) converge to the equilibrium $E_2^* = (2.1788, 0, -0.3426, 0.5855)$ when $\alpha = 0.95$, where $k_1 = 1$, $k_2 = 3$, $k_3 = 2$, $k_4 = 1$.

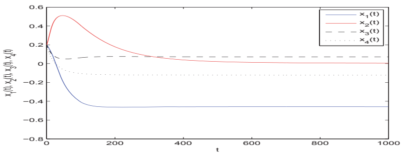

Figure 4: The trajectories of the controlled system (4) converge to the equilibrium $E_3^* = (-0.4589, 0, -0.0727, -0.1233)$ when $\alpha = 0.95$, where $k_1 = 2$, $k_2 = 1$, $k_3 = 2$, $k_4 = 3$.

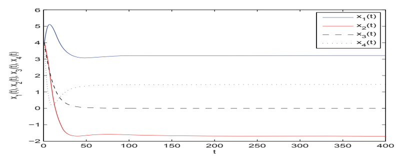

Figure 5: The trajectories of the controlled system (4) converge to the equilibrium $E_4^* = (3.2245, -1.7182, 0, 1.4657)$ when $\alpha = 0.95$, where $k_1 = 2$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$.

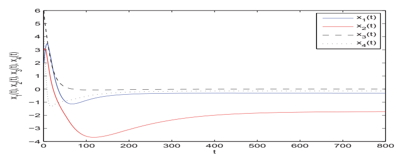

Figure 6: The trajectories of the controlled system (4) converge to the equilibrium $E_5^* = (-0.3155, -1.7182, 0, -0.1434)$ when $\alpha = 0.95$, where $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $k_4 = 1$.

6. Conclusion

In this paper, we have proposed a state feedback controller to stabilize fractional chaotic system to desirable equilibrium, and the equilibriums of the original system are kept completely. Analysis reveals that our control technique is feasible to implement and the obtained results are more simpler.

References

[1] S. Dadras, H. Momeni, “Control of a fractional-order economical system via sliding mode,” *Physica A,* vol.389, no.12, pp.2434–2442, 2010.

[2] Z. Wang, X. Huang, G.D. Shi, “Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay,” *Comput. Math. Appl,* vol.62, no.3, pp.1531–1539, 2011.

[3] M.F. Danca, R. Garrappa, W.K.S. Tangd, G.R Chen, “Sustaining stable dynamics of a fractional-order chaotic financial system by parameter switching,” *Comput. Math. Appl,* vol.66, no.5, pp.702–716, 2013.

[4] T.C. Earle, “Trust, confidence, and the 2008 global financial crisis,” *Risk Anal,* vol.29, no.6, pp.785–792, 2009.

[5] N. Hiltzik, “The new deal: a modern history,” *Simon & Schuster,* New York, 2011.

[6] B.G. Xin, J.Y. Zhang, “Finite-time stabilizing a fractional-order chaotic financial system with market confidence,” *Nonlinear Dyn,* vol.79, no.1, pp.1399–1409, 2015.

[7] I. Podlubny, “Functional differential equations,” *Academic Press,* New York, 1999.

[8] D. Matignon, “Stability results for fractional differential equations with applications to control processing,” *In: Proc. IMACS-SMC, Lille, France,* pp.963–968, 1996.

[9] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, “Equilibrium points, stability and numerical solutions of fractional order predator-prey and rabies models,” *J. Math. Anal. Appl,* vol.325, no.1, pp.542–553, 2007.

[10] A.E. Matouk, “Stability condition, hyperchaos and control in a novel fractional order hyperchaostic system,” *Phys. Lett. A,* vol.373, no,25, pp.2166–2173, 2009.

[11] K. Diethelm, N.J. Ford, A.D. Freed, “A predictor-corrector approach for the numerical solution of fractional differential equations,” *Nonlinear Dyn,* vol.29, no.1, pp.3–22, 2002.