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Synchronization of Distant Oscillators in a Ring Network

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Abstract—High-dimensional nonlinear phenomena in the field of natural sciences can often be explained as the result of a synchronization phenomenon of coupled oscillatory systems. In this study, we investigate how distant van der Pol oscillators coupled by resistors in a ring topology may synchronize. Using computer simulations, we find that distant oscillators can synchronize in-phase, anti-phase, and at $N/2$-phase (where N is the number of oscillators), in dependence of the oscillation frequency of adjacent oscillators.

1. Introduction

Synchronization phenomena can be observed in a wide range of fields such as physics, chemistry, biology, neuroscience, engineering, and so on. Synchronization may prove crucial in understanding - and mimicking in applications - decentralized information processing mechanisms, as found, in particular, in the living nature. Therefore, mathematical modeling studies of synchronization in oscillatory networks have become a topical issue [1]-[3]. As particularly nice examples, Endo et al. have presented the details of a theoretical analysis and corresponding circuit experiments on electrical circuits oscillators arranged in a ladder [4], a ring [5] and in a two-dimensional array topology [6]. Moreover, coupled oscillatory systems can also produce interesting phase patterns, including wave propagation, clustering, and complex phase patterns [7]-[10]. Seto et al. [11] have observed interesting synchronization phenomena when van der Pol oscillators with different frequencies are coupled by means of a resistor, in a star topology. This phenomenon is, however, restricted to small networks: Oscillators coupled in a star topology fail to synchronize if their number exceeds 4.

In our search for synchronization phenomena present in large-scale networks, we have therefore concentrated on van der Pol oscillators arranged in a ring topology, where each oscillator has its own oscillation frequency. For this arrangement, we found interesting nonlinear phenomena such as oscillation death, independent oscillations and double-mode oscillations [12]. Moreover, we have studied the interaction of van der Pol oscillators drawn from two sets of distinct oscillatory frequencies, where, in dependence of the different frequencies, we observed several interesting synchronization phenomena [12]. So far, the mechanisms of this type of group synchronization have, however, remained only partially understood.

In the present study, we focus on synchronization phenomena between distant van der Pol oscillators with different frequencies that are coupled by means of resistors in a ring topology. Using computer simulations, we find that the distant neighbors oscillators are synchronized in-phase, anti-phase or at $N/2$-phase, dependent on the oscillation frequency of the adjacent oscillators.

2. Circuit Model

The basic circuit model and the topology according to which the oscillators are arranged is shown in Fig. 1. In order to study the synchronization on this topology, we detune the frequency of each second oscillator by means of changing its capacitance, proceeding cyclically through the ring.

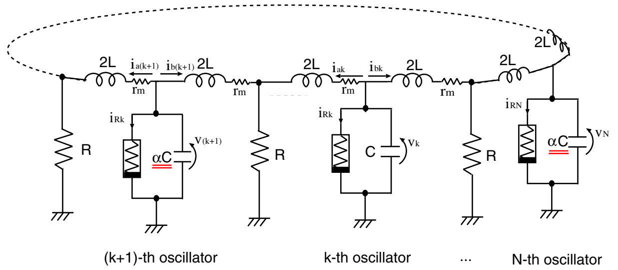

Figure 1: Ring of van der Pol oscillators with different frequencies.

We assume that the $v_k - i_{Rk}$ characteristics of the nonlinear resistor in each oscillator is given by the following third order polynomial equation

$$i_{Rk} = -g_1 v_k + g_3 {v_k}^3. \tag{1}$$

By changing the variables and the parameters,

$$v_k = \sqrt{\frac{g_1}{3g_3}} x_k, \quad i_k = \sqrt{\frac{g_1}{3g_3}} \sqrt{\frac{C}{L}} y_k, \quad t = \sqrt{LC}\tau,$$

$$\varepsilon = g_1\sqrt{\frac{L}{C}}, \quad \gamma = r\sqrt{\frac{C}{L}}, \quad \alpha = \frac{1}{\omega^2}, \quad \eta = r_m\sqrt{\frac{C}{L}},$$

the normalized circuit equations of the ring of oscillators finally read

$$\begin{cases} \dfrac{dx_k}{d\tau} = \omega_k^2 \varepsilon x_k(1 - {x_N}^2) - \omega_k^2(y_{aN} + y_{bN}) \\ \dfrac{dy_{ak}}{d\tau} = \dfrac{1}{2}x_k - \eta y_{ak} - \gamma(y_{ak)} + y_{b(k+1)}) \\ \dfrac{dy_{bk}}{d\tau} = \dfrac{1}{2}x_k - \eta y_{bk} - \gamma(y_{a(k-1)} + y_{bk)}) \\ \qquad\qquad (k = 1, 2, \cdots, N), \end{cases} \tag{2}$$

where

$$y_{a0} = y_{aN}, \qquad y_{b(N+1)} = y_{b1}. \tag{3}$$

In this system of equations, ω_k denotes the frequency of the kth oscillator, γ corresponds to the coupling strength and ε captures the nonlinearity of the oscillators. For the simulations, we evaluated Eq. (2) using a fourth-order Runge-Kutta method. For this setting, we obtained results as shown and discussed in the following paragraph.

3. Synchronization phenomena when each second oscillator is detuned

3.1. N=4

In this case, the two types of oscillators with detunable frequencies are placed alternately, as is shown in Fig. 2, where α denotes the from the standard frequency $\omega = 1.0$ detuned frequency. We now focus on the synchronization between oscillator 1 and oscillator 3, which have the standard frequency ($\omega = 1.0$), when the frequencies oscillator 2 and oscillator 4 are periodically changed to $\omega = \alpha$.

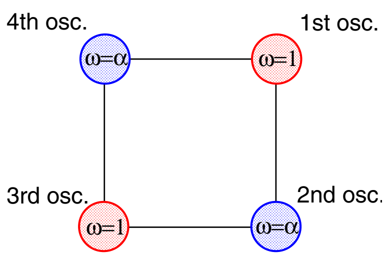

Figure 2: Model circuit for $N = 4$.

The obtained results are collected in Figs. 3 and 4. In these figures, the top rows show the attractors obtained from oscillator 1 and oscillator 3, the middle row displays the phase difference between the ith and the $(i+2)$th oscillator, whereas the bottom row shows the time wave-forms. Figure 3 has been obtained for $\alpha = 0.64$. In this case, the oscillators 1 and 3 are almost synchronized in-phase. The result obtained for $\alpha = 3.47$ is shown in Fig. 4. Now, oscillator 1 and oscillator 3 oscillators are almost synchronized anti-phase, whereas oscillator 2 does not synchronize with oscillator 4 at all.

Next, we calculated the phase difference between oscillator 1 and oscillator 3 if the frequencies α of the oscillators 2 and 4 are increased from 0.1 to 3.5. The results shown in Fig. 5 demonstrate that when α is smaller than 0.5, non-synchronization is observed. Around $\alpha \simeq 1.0$, in-phase synchronization is obtained. By further increasing the value of α, anti-phase synchronization can be triggered.

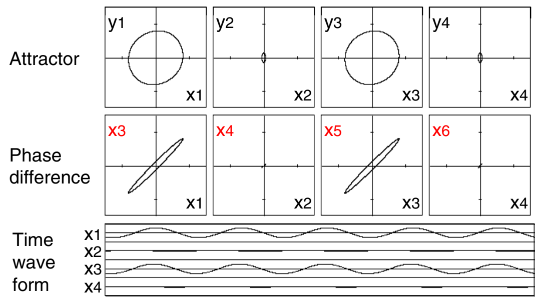

Figure 3: Synchronization results for $N = 4$ ($\alpha = 0.64$).

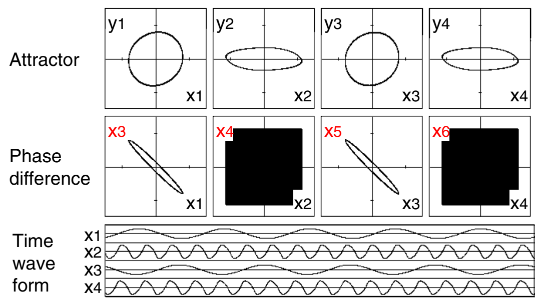

Figure 4: Synchronization results for $N = 4$ ($\alpha = 3.47$).

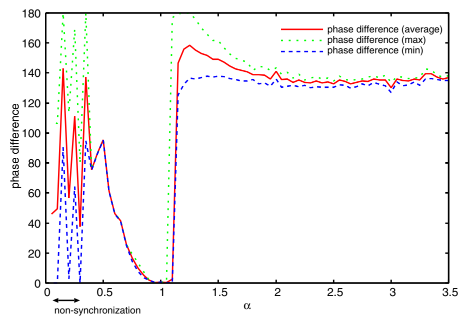

Figure 5: Phase difference between x_1 and x_3, in dependence of the frequencies of x_2 and x_4.

3.2. N=6

Next, we consider a circuit model which is composed of six oscillators, as shown in Fig. 6. The observed synchronization phenomena for $\alpha = 0.64$ are shown in Fig. 7. From this figure, we infer that almost the same synchronization phenomena as those observed for $N = 4$ emerge. Namely, that any ith oscillator is almost in-phase synchronized with the $(i + 2)$th oscillators, at the two oscillators' inherent frequency.This synchronization pattern changes at $\alpha = 3.47$. In this case, the three oscillators with the standard frequency are synchronized at 120 degree phase difference (see Figs. 8,9).

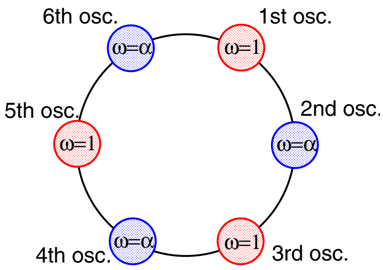

Figure 6: Circuit model for $N = 6$.

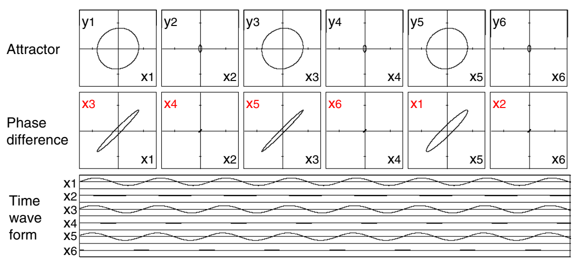

Figure 7: Synchronization results for $N = 6$ (at α=0.64).

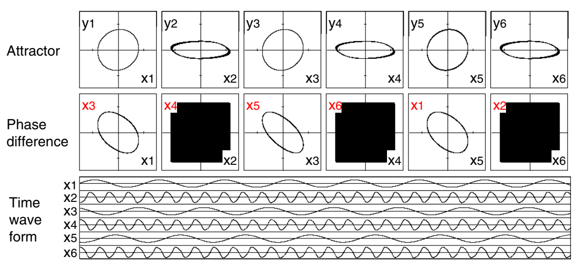

Figure 8: Synchronization results for $N = 6$ (at α=3.47).

3.3. N=8, 10

Furthermore, we investigated what synchronization patterns emerge if the number of coupled oscillators is set to

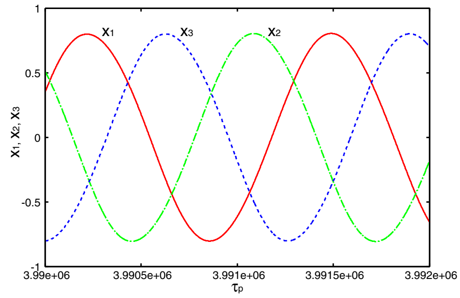

Figure 9: Synchronization at 120 degree difference (α=3.47).

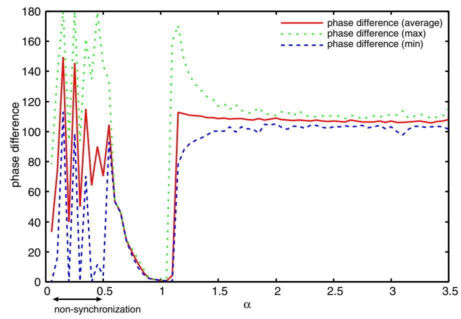

Figure 10: Phase difference between x_1 and x_3 in dependence of the frequencies of x_2, x_4 and x_6.

8 and 10, respectively. In the both cases, if we choose $\alpha = 0.64$, we observed that, as before for $N = 4, 6$, the ith and the $(i + 2)$ oscillators synchronize in-phase. When frequency α is fixed to $\alpha = 3.47$, however, 2 pairs of anti-phase synchronized distant neighbors emerge for $N = 8$ (see Fig. 11), and 5-phase synchronization emerges for $N = 10$ (see Fig. 12).

From these results, we infer that $N/2$ pairs of anti-phase synchronized oscillators could be observed if half of the total number of coupled oscillator is even. If the number of pairs $N/2$ is an odd number, we, however, expect $N/2$-phase synchronization to occur (at least for a not too high number N of oscillators).

4. Synchronization phenomena when two out of three oscillators are detuned

The conceptual circuit model corresponding to this situation is shown in Fig. 13. Figure 14 shows the simulation results if α is set to $\alpha = 0.64$. In this case, we observe non-synchronization between oscillator 1 and oscillator 4, although they have the same standard oscillation frequency. Instead, neighboring oscillators that have the changing frequencies (2nd-3rd, and 5th-6th oscillators) are synchronized anti-phase, as is shown in Fig. 14. At variance with the results presented in the previous section, the

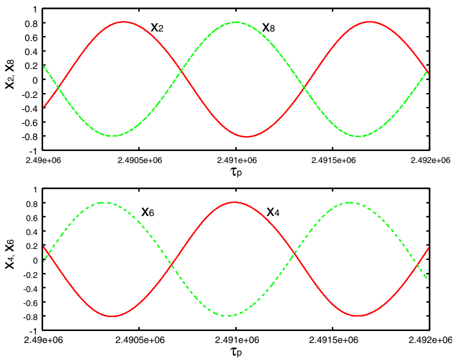

Figure 11: 2 pair anti-phase synchronization.

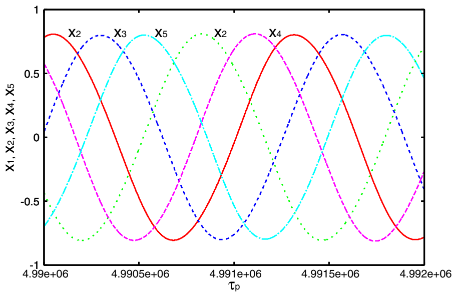

Figure 12: 5-phase synchronization.

oscillators with $\omega = 1$ can now oscillate even at the lower value $\alpha = 0.64$, since seemingly they can benefit from the neighbor that is in the same situation. Together, they can make it.

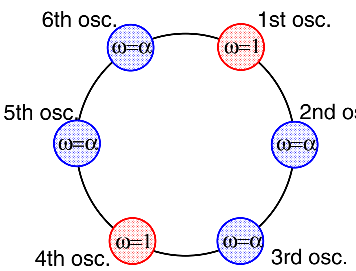

Figure 13: Conceptual circuit model (N=6).

5. Conclusions

In this study, we have investigated synchronization phenomena of distant van der Pol oscillators on a ring topology, where we cycle through the topology, detuning oscillators in a regular fashion. Using computer simulations, we have confirmed that the distant oscillators can synchronize in-phase, anti-phase, and $N/2$-phase, in dependence of the oscillation frequency of the adjacent oscillators.

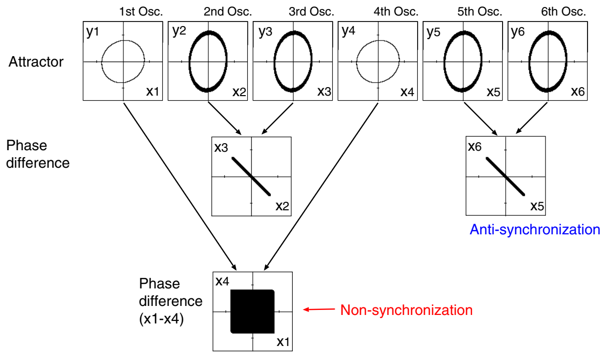

Figure 14: Anti-phase synchronization and non-synchronization ($\alpha = 0.64$).

References

[1] M. G. Rosenblum, A. S. Pikovsky and J. Kurths, "From phase to lag synchronization in coupled chaotic oscillators," *Phys. Rev. Let.*, vol.78, no.22, pp.4193-4196, Jun. 1997.

[2] Z. Zheng, G. Hu and B. Hu, "Phase slips and phase synchronization of coupled oscillators," *Phys. Rev. Let.*, vol.81, no.24, pp.5318-5321, Dec. 1998.

[3] C. Zhou, J. Kurths, I. Z. Kiss and J. L. Hudson, "Noise-enhanced phase synchronization of chaotic oscillators," *Phys. Rev. Let.*, vol.89, no.1, (014101), Jul. 2002.

[4] T. Endo and S. Mori, "Mode analysis of a multimode ladder oscillator," *IEEE Trans. Circuits Syst.*, vol.23, pp.100-113, Feb. 1976.

[5] T. Endo and S. Mori, "Mode analysis of two-dimensional low-pass multimode oscillator," *IEEE Trans. Circuits Syst.*, vol.23, pp.517-530, Sep. 1976.

[6] T. Endo and S. Mori, "Mode analysis of a ring of a large number of mutually coupled van der Pol oscillators," *IEEE Trans. Circuits Syst.*, vol.25, no.1, pp.7-18, Jan. 1978.

[7] Y. Nishio and S. Mori, "Mutually coupled oscillators with an extremely large number of steady states," *Proc. of ISCAS'92*, vol.2, pp.819-822, May 1992.

[8] L. Pivka, "Autowaves and spatio-temporal chaos in CNNs Part I: A tutorial," *IEEE Trans. Circuits Syst. I*, vol.42, pp.638-649, Oct. 1995.

[9] M. Yamauchi, Y. Nishio and A. Ushida, "Phase-waves in a ladder of oscillators" *IEICE Trans. Fundamentals*, vol.E86-A, no.4, pp.891-899, Apr. 2003.

[10] Y. Uwate and Y. Nishio, "Complex Phase Synchronization in an Array of Oscillators Coupled by Time-Varying Resistor," *Porc. of IJCNN'06*, pp. 8345-8350, Jul. 2006.

[11] Y. Setou, Y. Nishio and A. Ushida, "Synchronization Phenomena in Resistively Coupled Oscillators with Different Frequencies," *IEICE Trans. Fundamentals*, vol.E79-A, no.10, pp.1575-1580, Oct. 1996.

[12] Y. Uwate, Y. Nishio and R. Stoop, "Group Synchronization of van der Pol Oscillators with Different Frequencies," *Porc. of NOLTA'08*, pp. 612-615, Sep. 2008.