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Synchronization in a ring of weakly coupled Rössler oscillators

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Abstract— This paper studies dynamics of a ring of weakly coupled Rössler oscillators. The chaotic orbits of the coupled oscillators are stabilized on periodic orbits which are slightly different from unstable periodic orbits embedded within a single Rössler oscillator. We observe spatial phase patterns on the stabilized orbits in the coupled oscillators. Furthermore, it is confirmed that the stabilization remains even if with a small-world topology.

1. Introduction

Synchronization is observed everywhere in the natural world [1]. Understanding the synchronization in simple coupled oscillators is considerable significant [2, 3]. The synchronization occurs not only in coupled periodic oscillators but also in chaotic oscillators. It is well known that this phenomenon depends on a coupling strength, parameters of the oscillators, and a network topology [4]. Recently, the synchronization in coupled oscillators on complex network topologies, such as small-world and scale-free type, have been studied [4–6].

Iwase *et al.* reported that a weak coupling can induce periodic orbits in coupled Rössler oscillators on a scale-free topology (i.e., Dorogovtsev-model) [9, 10]. Zhan *et al.* showed that, in a ring of weakly coupled chaotic oscillators, the chaotic orbits are stabilized on unstable periodic orbits, which are embedded within a chaotic attractor [7, 8]. This phenomenon can be regarded as one of the methods for controlling chaos [11] or a new type of amplitude death [12] [1]. Although this phenomenon would be useful in a situation, where one wants to stabilize chaotic oscillations in weakly coupled real systems, to our knowledge, there have been few effort to investigate it.

The present paper investigates such phenomenon numerically, and answers the following questions.

- Are the stabilized periodic orbits induced by a weak coupling identical to unstable periodic orbits embedded within a single Rössler oscillator?
- Is there a rule in phase patterns on the stabilized orbits?
- Does the stabilization remains even if with a small-world topology?

[1]Amplitude death is well known as a stabilization of unstable fixed points induced by mutual interactions among two or more oscillators.

2. Coupled chaotic oscillators

A network is composed of nodes and their edges: oscillators and their connections are regarded as nodes and edges respectively. The present paper focuses on a network of Rössler oscillators. The oscillator at node i is described by three variables x_i, y_i, z_i and coupling signal u_i,

$$\begin{cases} \dot{x}_i = -wy_i - z_i + \epsilon u_i \\ \dot{y}_i = wx_i + ay_i \\ \dot{z}_i = b + z_i(x_i - c) \end{cases} \quad (i = 1, 2, \ldots, N), \tag{1}$$

$$u_i = \sum_{j=1}^{N} G_{ij} x_j, \tag{2}$$

where the parameters are fixed as $w = 0.99, a = 0.165, b = 0.2, c = 10.0$ and $0 < \epsilon \ll 1$ is the coupling strength [7]. A single Rössler oscillator with these parameters behaves chaotically. G_{ij} is the element of the coupling matrix $\mathbf{G}$, where $G_{ij} = G_{ji} = 1$ if nodes i and j are coupled and $G_{ij} = G_{ji} = 0$ otherwise. The diagonal elements of $\mathbf{G}$ are defined by $G_{ii} = -\sum_{j=1}^{N} G_{ij}$. In a ring of coupled oscillators (see Fig. 1), $\mathbf{G}$ is written by

$$\mathbf{G} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 & -2 \end{pmatrix}. \tag{3}$$

Let us review previous work [7]. Figure 2(a) shows the orbit of a single chaotic Rössler oscillator without coupling ($\epsilon = 0$). The Rössler oscillators are coupled by extremely weak connection ($\epsilon = 3.0 \times 10^{-4}$) on the ring topology as shown in Fig. 1. Figure 2(b) indicate that chaotic motions

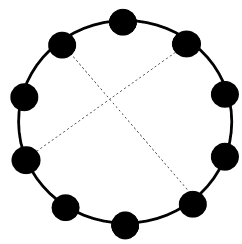
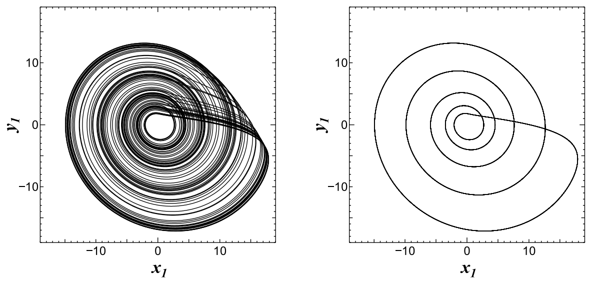


Figure 1: Ring of coupled oscillators ($N = 10$)



(a) $\epsilon = 0$ (b) $\epsilon = 3.0 \times 10^{-4}$

Figure 2: Orbits of Rössler oscillators ($N = 10$)

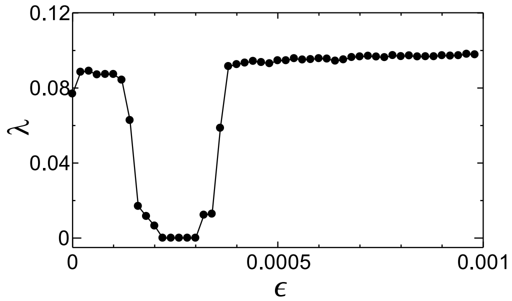


Figure 3: Largest Lyapunov exponent λ against ϵ (ring type, $N = 10$)

in all the oscillators are stabilized on an unstable period-5 orbit. Figure 3 shows the largest Lyapunov exponent λ against $\epsilon \in [0, 10.0 \times 10^{-4}]$ [2]. It can be seen that $\lambda \simeq 0.09$ is maintained in the range of $\epsilon \in [0, 1.5 \times 10^{-4}]$ or $\epsilon > 3.5 \times 10^{-4}$, where all the oscillators are chaotic. On the other hand, in the range of $\epsilon \in [2.0 \times 10^{-4}, 3.0 \times 10^{-4}]$, all the oscillators are periodic (i.e., $\lambda \simeq 0$) as shown in Fig. 2(b). The phenomenon shown in Fig. 3 are observed for arbitrary N oscillators [7] [3]. In addition, Zhan *et al.* suggested that the stabilized periodic orbit of Fig. 2(b) is identical to the unstable periodic orbit embedded within the single Rössler oscillator (Fig. 2(a)).

3. Periodic orbits induced by extremely weak coupling

This section investigates the numerical results suggested in reference [7]. The unstable period-5 orbit embedded within the single Rössler oscillator is extracted numerically (see Fig. 4). Since the orbits in Fig. 4 and Fig. 2(b) look like same, the suggestion in reference [7] seems totally natural. If the suggestion is correct, the coupling signals of all the oscillators should vanish, $u_i = 0, \forall i \in \{1, \ldots, N\}$, after transient period. [4]

[2] λ is estimated by averaging in time $t \in [2.5 \times 10^5, 3.0 \times 10^5]$ with time step $\Delta t = 0.01$.

[3] We also confirmed this phenomenon for $N = 2 \sim 100$.

[4] The control signals for the controlling chaos and the coupling signals for the amplitude death vanish when the stabilization is achieved.

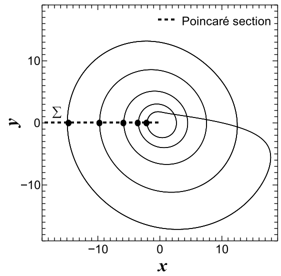
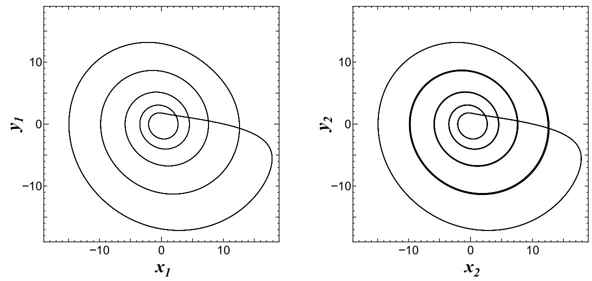


Figure 4: Unstable period-5 orbit and Poincaré section



(a) Oscillator 1 (b) Oscillator 2

Figure 5: Stabilized orbit in coupled two Rössler oscillators ($\epsilon = 3.0 \times 10^{-4}, N = 2$)

To check the suggestion, we consider the simplest case ($N = 2$). The orbits of two oscillators after coupling are shown in Figs. 5(a) and (b). As a matter of course, these orbits also resemble the unstable periodic orbit in Fig. 4. The coupling signals $u_{1,2}$ after transient period are shown in Fig. 6. If the suggestion were correct, these signals should vanish. However, the signals $u_{1,2}$ are always added to the oscillators: they do not synchronize completely.

The difference of these orbits on a Poincaré section is estimated. First of all, the following Poincaré section is defined (dashed line in Fig. 4):

$$\Sigma := \{(x, y, z) : y = 0, x < 0\}. \quad (4)$$

The unstable period-5 orbit in Fig. 4 intersects with Σ at the points

$$\overline{\mathbf{X}}^*(k) := \begin{bmatrix} \overline{x}^*(k) & 0 & \overline{z}^*(k) \end{bmatrix}^T, \quad k = 1, \ldots, 5. \quad (5)$$

$\overline{x}^*(k)$ are defined as follows:

$$\overline{x}^*(1) < \overline{x}^*(2) < \overline{x}^*(3) < \overline{x}^*(4) < \overline{x}^*(5) < 0.$$

In the same way, the points where the periodic orbit of coupled oscillator i intersects with Σ are given by

$$\overline{\mathbf{X}}_i(k) := \begin{bmatrix} \overline{x}_i(k) & 0 & \overline{z}_i(k) \end{bmatrix}^T, \quad k = 1, \ldots, 5. \quad (6)$$

where the following order is also defined:

$$\overline{x}_i(1) < \overline{x}_i(2) < \overline{x}_i(3) < \overline{x}_i(4) < \overline{x}_i(5).$$

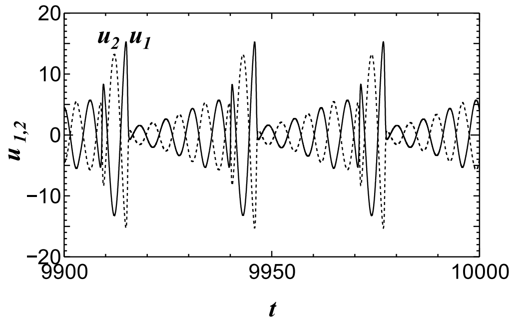


Figure 6: Coupling signals u_i ($N = 2$)

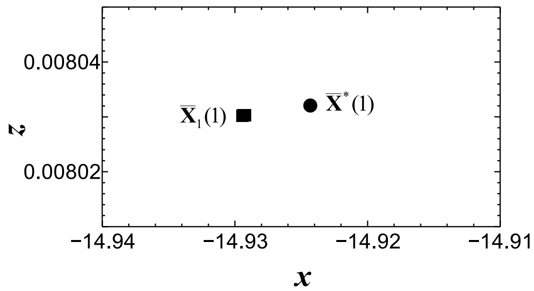


Figure 7: $\overline{\mathbf{X}}^*(1)$ and $\overline{\mathbf{X}}_1(1)$ on section Σ

Figure 7 shows $\overline{\mathbf{X}}^*(1)$ and $\overline{\mathbf{X}}_1(1)$, which are obviously separated. It is confirmed numerically that $\|\overline{\mathbf{X}}^*(k) - \overline{\mathbf{X}}_{1,2}(k)\| > 0.001, \forall k \in \{1, \ldots, 5\}$. This result concludes that the suggestion in reference [7] is not strictly correct. However, it seems that the difference does not matter in a situation, where one can tolerate margins of such error.

4. Spatial phase patterns

In this section, we consider the spatial phase patterns in the coupled oscillators.

Figure 8 shows the time series data of x_i for coupled four oscillators ($N = 4$). Although all the oscillators settle on a period-5 orbit, their amplitudes are different at any time. In addition, all the oscillators intersect with $x_i = 0$ almost at the same time. These facts show that the phases of all the oscillators are locked. The phase locked pattern depends on the initial condition.

Now we investigate the patterns in detail. Let us assume the following situation: the orbits of all the oscillators $\overline{\mathbf{X}}_i(k)$ ($k = 1, \ldots, 5$) are stabilized near the unstable period-5 orbit $\overline{\mathbf{X}}^*(k)$ ($k = 1, \ldots, 5$). Furthermore, we assume that the orbit of oscillator i on Σ at $t = t_0$, $x_i(t_0)$, is located at $x_i(t_0) \simeq \overline{x}^*(1)$. The nearest neighbors of oscillator i at $t = t_0$ are located at $x_{i-1}(t_0) \simeq \overline{x}^*(m)$ and $x_{i+1}(t_0) \simeq \overline{x}^*(n)$ respectively, where $m \in \{1, \ldots, 5\}$ and $n \in \{1, \ldots, 5\}$ depend on the initial conditions. This pattern is defined as (m 1 n). Our numerical simulations provide the frequency distribution of the following 15 patterns:

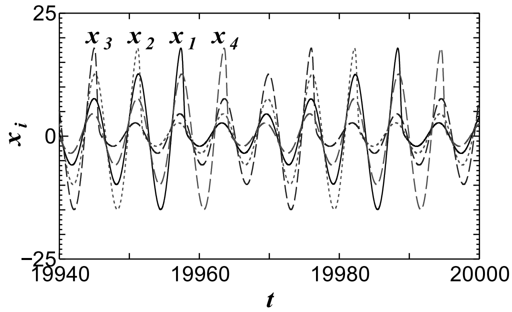


Figure 8: Time series data x_i ($i = 1, \ldots, 4$). ($\epsilon = 3.0 \times 10^{-4}, N = 4$)

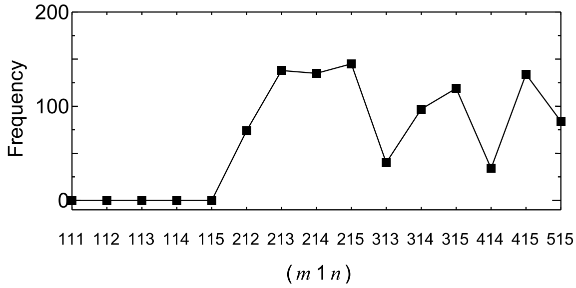


Figure 9: Frequency distribution of pattern (m 1 n) ($\epsilon = 3.0 \times 10^{-4}, N = 10$).

$$\begin{aligned} &(111), (112), (113), (114), (115), \\ &(212), (213), (214), (215), (313), \\ &(314), (315), (414), (415), (515). \end{aligned} \tag{7}$$

Since the ring topology described by Eq.(3) is symmetry, the permutation of m and n can be ignored. For example, the pattern (3 1 2) is treated as the pattern (2 1 3).

The frequency distribution of pattern (m 1 n) is estimated by the algorithm given in appendix A. Figure 9 shows the frequency distribution for $N = 10$. We have confirmed that all the frequency distributions for $N = 4 \sim 10$ are almost the same as Fig. 9 qualitatively. It must be emphasized that the pattern (111), (112), (113), (114), (115) never occur. This fact implies that the nearest neighbor oscillators do not synchronize.

5. Small-world topology

We investigate whether the stabilization in the coupled oscillators ($N = 10$) remains even if with a small-world topology. The shortcuts, the dotted lines in Fig. 1, are added to the network. The $2l$ oscillators, which are not nearest neighborhood on the ring, are chosen at random and the l shortcuts are added. The largest Lyapunov exponent λ is estimated against $\epsilon \in [0, 10.0 \times 10^{-4}]$ five times depending on random l shortcuts. Figure 10 shows λ against the coupling strength ϵ, where λ is the average value. Although the range ($\lambda \simeq 0$) is narrow in Fig. 10 compared with Fig.

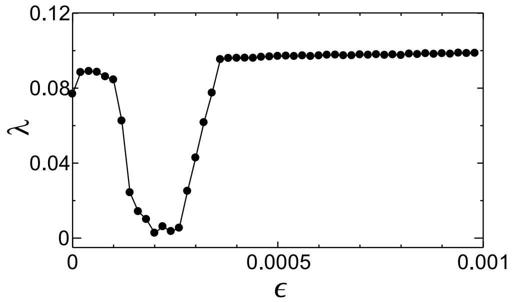


Figure 10: Largest Lyapunov exponent λ against ϵ (small world type, $N = 10, l = 3$)

3, all the oscillators are periodic. The stabilization can be confirmed even with a small-world topology.

6. Conclusion

This paper investigated the synchronization in a ring of weakly coupled Rössler oscillators. The main results are summarized below.

- The stabilized periodic orbits are slightly different from the unstable periodic orbit embedded within the single Rössler oscillator.
- Several spatial phase patterns on the stabilized orbits occur in the coupled oscillators.
- Stabilization remains even if with a small-world topology.

A. Algorithm to estimate the frequency

The frequency distribution is estimated by the following procedures. 1) The initial values of the oscillators are chosen at random on a single chaotic attractor; 2) If the following conditions, (a) and (b), are satisfied for $t < 2 \times 10^4$, one proceed to the next step, otherwise go back to step 1);

Condition (a) $\lambda < 0.001$

Condition (b) $\overline{x}_i(t) \simeq \overline{x}^*(k_i), k_i \in \{1, \ldots, 5\}, \forall i \in \{1, \ldots, N\}$

3) The frequency distribution of each situation is accumulated for N oscillators judging by the pattern of (7). 4) If the total of the frequency is less than 1000, one proceed to step 1), and otherwise this estimation is finished. The condition (a) shows that the orbits of all the oscillators are almost settled down on the periodic orbits. The condition (b) shows that the phase of all the oscillators is locked. Remark that one cannot proceed from step 2) to step 3) depending on the initial values. The probability that one can proceed is about 5% in our simulation.

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