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Analyses of the dynamics of interconnected van der pol models based-on a concept of potential with active areas

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Abstract—We present various bursting wave forms that are obtain from a simple model of Hodgkin-Huxley type. The model is a typical example whose characteristics can be discussed through a concept of potential with active areas. A potential function is able to provide a global landscape for dynamics of a model, and the dynamics are explained in relation to the disposition of the active area on the potential. We obtain the potential functions and the active areas for a Hindmarsh-Rose model,a Morris-Lecar system, and a Hodgkin-Huxley system, and hence we are able to discuss the common properties among these models based on the concept of potential with active areas. Furthermore, we are able to understand intuitively a bifurcation of an interconnected van der pol system by using the potential, so that the new concept is very useful to describe the dynamics of interconnected systems.

1. Introduction

We have proposed the Inverse function Delayed model(ID model) as one of neural networks[1]. This model has physiological well-grounded negative resistance in its dynamics, we have reported that this model substantially grows in performance in various intelligent information processings[2][3]. In addition, we consider that the burst firing oscillation has prospects of capabilities of effective tool for information processings consequently. We proposed Burst ID model of Hodgkin-Huxley(H-H) type that has burst firing oscillation characteristics[4], we have proposed that we are able to explain the characteristics and various wave forms as a motion of particulars on the potential with the active area. The potential function gives global dynamic characteristics, and this dynamics are explained in relation to the disposition of the active area on the potential. This property is applied to the other neuron models including chaotic dynamics, and we obtain the potential function and active areas for Hindmarsh-Rose model, Morris-Lecar model and so on[5]. In this paper, we apply this concept to van der pol model and analyze the interconnection system with this model by using the potential, expecting that such a new concept is also very useful to describe the dynamics of interconnected systems.

2. Potential and Active areas of Burst Inverse function Delayed model

We have made the ID model burst to add the third variable z corresponded to variable m in H-H model. This model is expressed as the following equations

$$\tau_x \frac{dx}{dt} = u + \gamma z - g(x) \tag{1}$$

$$\tau_u \frac{du}{dt} = Wx - u \tag{2}$$

$$\tau_z \frac{dz}{dt} = -z + z^{\infty}(x) + \frac{1}{\gamma}\theta, \tag{3}$$

where $\tau_u >> \tau_z \geq \tau_x$,

$$g(x) = 1/8\log(x/(1-x)) - 2(x-0.5) \tag{4}$$

$$z^{\infty}(x) = \tanh\{4(x-0.7)\}. \tag{5}$$

Equations (1), (2) and (3) can be transformed into the Eq. (6) of one variable by deleting z and u

$$\begin{aligned}
&\frac{d^3x}{dt^3} + (\eta(x) + \frac{1}{\tau_u})\frac{d^2x}{dt} + \{\frac{d\eta(x)}{dx}\frac{dx}{dt} + \\
&\frac{1}{\tau_x\tau_z}(\frac{dg(x)}{dx} - \gamma\frac{dz^{\infty}(x)}{dx} - \frac{\tau_z}{\tau_u}W) + \frac{1}{\tau_u}\eta(x)\}\frac{dx}{dt} \\
&= F(x,\theta) \\
&= -\frac{\partial U(x,\theta)}{\partial x},
\end{aligned} \tag{6}$$

where $\eta(x) = \frac{1}{\tau_x}\frac{dg(x)}{dx} + \frac{1}{\tau_z}$ and $U(x,\theta)$ is a kind of the potential. The equilibrium point x_0 depends on the external input θ, and it is calculated by using the equation of $\partial U(x,\theta)/\partial x = 0$. We have obtained the characteristic equation and tried to analyze the stability at neighborhood of a equilibrium point according to Hurwitz's theorem. In this way, we obtain Eqs. (7)∼(11), in case which these functions are positive, the system stays in a stable state. Equation (7) describes the curvature factor of the potential, therefore the system is unstable if it is a negative value. In other words, we identify areas that Eqs.(8)∼(11) is negative as the active areas.

$$b_0(x) = \frac{\partial^2 U(x_0,\theta)}{\partial x^2} \tag{7}$$

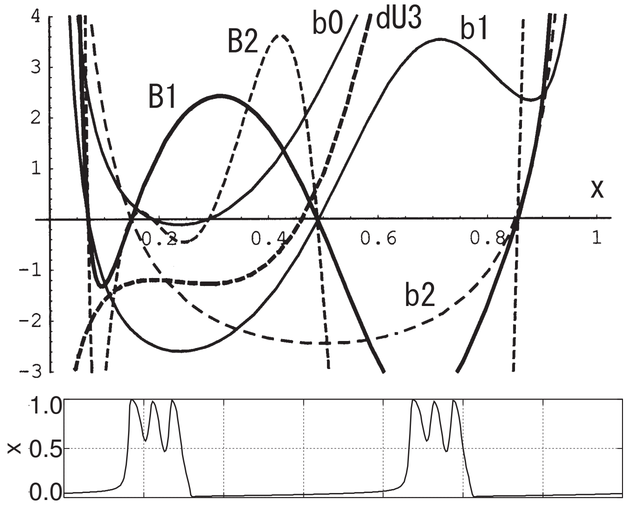

Figure 1: Burst ID active areas, potential and waveform

$$b_1(x) = \frac{1}{\tau_x\tau_z}\{\frac{dg(x_0)}{dx} - \gamma\frac{dz^\infty(x_0)}{dx} - \frac{\tau_z}{\tau_u}W\} + \frac{\eta(x_0)}{\tau_u} \quad (8)$$

$$b_2(x_0) = \eta(x_0) + \frac{1}{\tau_u} \quad (9)$$

$$B_1(x_0) = b_2(x_0)b_1(x_0) - b_0(x_0) \quad (10)$$

$$B_2(x_0) = b_0(x_0)B_1(x_0). \quad (11)$$

Figure 1 shows the differentiation of the potential function $\partial U/\partial t$, active areas, $b_0(x)$, $b_1(x)$, $b_2(x)$, $B_1(x)$ and $B_2(x)$, the output does not diverge, because potential function is reentrant and active areas are localized. $b_1(x)$ active area and $b_2(x)$ have overlaps of each active areas partly, the former causes the output to oscillate slowly, in other hand, the later creates fast oscillations. Furthermore, chaotic dynamics are observed from $\theta = 3.0$ to $\theta = 4.0$.

3. van der Pol model

3.1. Basic equations

The van der Pol system is expressed as following equation

$$\frac{dx^2}{dt^2} - q(1 - x^2)\frac{dx}{dt} = -x. \quad (12)$$

We rewrite this equation Eq.(13), because we can set the negative resistance region arbitrarily considering the external input.

$$\frac{d^2x}{dt^2} + \epsilon\{(x-\alpha)^2 - \beta\}\frac{dx}{dt} = Wx + \omega_{ij}y + \theta. \quad (13)$$

Equation (13) can be transformed into the following equation of two variables.

$$\frac{dx}{dt} = u - \epsilon(\frac{1}{3}x^3 - \alpha x^2 - \beta x) \quad (14)$$

$$\frac{du}{dt} = Wx + \omega_{ij} + \theta. \quad (15)$$

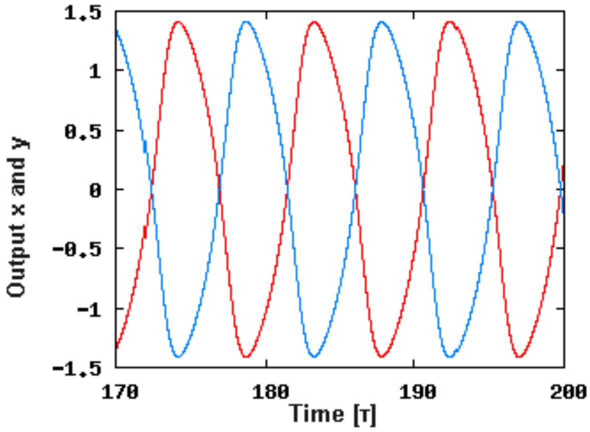

Figure 2: Time series of the output $x(t)$ and $y(t)$ with $W = -1.0$, $\omega = -0.5$, α=0, β=0.49 and ϵ=1.0. In cases which the active areas are symmetrical to the Y axis(Fig.4), these units show behavior of synchronized oscillation in phase or opposite phase depended on the initial values.

Where $x(y)$, u, W, θ and, ω_{ij} are the output of the unit, the internal state, the self-connection, the baias, and the connection weight from the unit j, thus the range of the negative resistance region is $|x - \alpha| < \sqrt{\beta}$.
We have interconnected two units expressed by Eq.(13), however the interconnection is equal($w_{ij} = w_{ji} = w$) and out of consideration of the bias θ.

$$\frac{d^2x}{dt^2} + \epsilon\{(x-\alpha)^2 - \beta\}\frac{dx}{dt} = Wx + \omega y$$
$$\frac{d^2y}{dt^2} + \epsilon\{(y-\alpha)^2 - \beta\}\frac{dy}{dt} = Wy + \omega x \quad (16)$$

In case which units are interconnected, the output $x(t)$, $y(t)$ are shown in Fig. 2. Equation (16) can be transformed into the one-variable equation

$$\frac{d^4x}{dt^4} + b_3(\ddot{x}, \dot{x}, x)\frac{d^3x}{dt^3} + b_2(\ddot{x}, \dot{x}, x)\frac{d^2x}{dt^2} + b_1(\dot{x}, x)\frac{dx}{dt}$$
$$= -(W^2 - \omega^2)x$$
$$= -F(x)$$
$$= -\frac{\partial U(x)}{\partial x}, \quad (17)$$

thus we can obtain the potential function

$$U(x) = \frac{1}{2}(W^2 - \omega^2)x^2. \quad (18)$$

This potential function becomes the convex function if $W < \omega$, and we obtain the divergence of the output x(Fig. 3). If $W > \omega$, the potential function forms the concave function and we can avoid the divergence. There are active areas, so this system spontaneously oscillates in continuity, and we can make sure it by numerical experimentations.

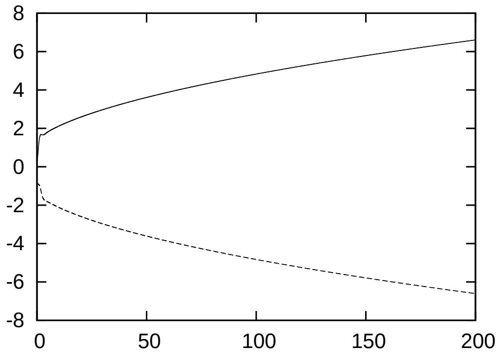

Figure 3: Time series of the output x(t) and y(t) with $W = -1.0$, $\omega = -1.1$, α=0, β=0.49,ϵ=1. The potential function is the convex function, so the output diverge for infinity.

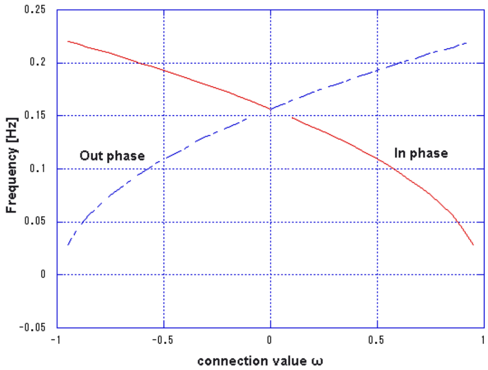

Figure 4: The frequency characteristics of inphase and antiphase oscillations, when $\alpha = 0.0, \beta = 0.49, \epsilon = 1.0$ *and* $W = -1.0$.

3.2. characteristics of interconnected van der Pol models for $\alpha = 0$

In this subsection, we show some characteristics of the interconnected van der Pol models, the amplitude and frequency. When $\alpha = 0$, the disposition of active areas is symmetrical, we can obtain two different solution orbits, inphase and antiphase oscillation. If connection value is week, the oscillation state is stable at one side. When the connection value is positive, the in-phase oscillation is not stable. Reverse case, the antiphase oscillation is unstable, so, these characteristics is not continuous value nearby the point of origin. These oscillations have the different characteristic features. Figure 4 shows the frequency characteristics with changing the connection value ω. The frequency of antiphase oscillation increases with increasing connection value, in contrast, the frequency of in-phase oscillation decreases.

The characteristics of oscillation amplitude is showed in Fig.5. The amplitude has a value about 1.4. When the absolute values of amplitude is about 0.7, the amplitude value

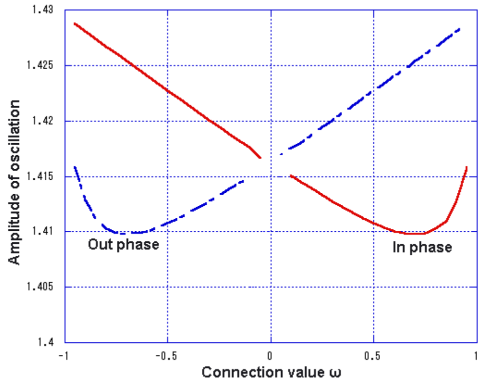

Figure 5: The amplitude characteristics of inphase and antiphase oscillations,when $\alpha = 0.0, \beta = 0.49, \epsilon = 1.0$ and $W = -1.0$.

take an extreme value.

3.3. Active areas

The equilibrium point x_0 is obtained according to the potential function, we also obtain the following equations for Burst ID model,

$$b_0(x) = W^2 - \omega^2 \tag{19}$$
$$b_1(x) = -W\{A(x) + C(x)\} \tag{20}$$
$$b_2(x) = -2W + A(x)C(x) \tag{21}$$
$$b_3(x) = \{A(x) + C(x)\} \tag{22}$$
$$B_1(x) = b_2(x)b_3(x) - b_1(x) \tag{23}$$
$$B_2(x) = B_1(x)b_1(x) - b_3(x)^2 b_0(x), \tag{24}$$

where, $A(x) = \epsilon\{(x-\alpha)^2 - \beta\}, C(x) = \epsilon\{(\frac{W}{\omega}x+\alpha)^2 - \beta\}$. If all these equations are positive, the system is in a stable state. In contrast, if just one equation is negative, it is unstable. We consider that, inside of the active areas, the systems are subjected to force and these wave forms show changes.

3.4. Parameter dependency of active areas

We are able to set up the active areas with the control parameters. The active areas are symmetrical to the Y axis, if $\alpha = 0$. Figure 6 shows the width of $b_1(x)$ active area($\alpha = 0.0, \beta = 0.49, \epsilon = 1.0$ and $W = -1.0$) as a function of parameter ω. $b_1(x)$ active area grows wider with increasing ω. This function represents the resistance of a moving particle in the potential, we consider that this active area controls the frequency of oscillations. $b_1(x)$ is equal function with $b_3(x)$, if self-connection $W = -1$ and each parameters is the same value, and hense $b_3(x)$ active area has the similar characteristics to $b_1(x)$.
Figure (7) shows the width of $b_2(x)$ active area as a function of ω. $b_2(x)$ active area disappear with decreasing ω. $b_2(x)$ function represents the mass of a moving particle,

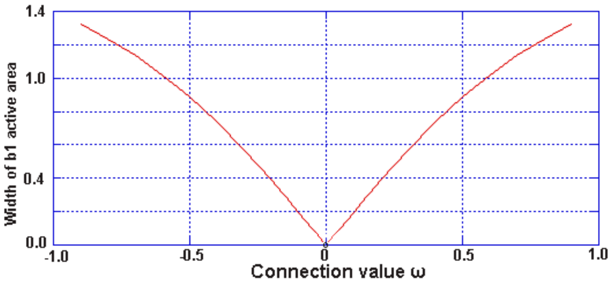

Figure 6: Dependency of width of $b_1(x)$ active area on the parameter ω with $\alpha = 0.0, \beta = 0.49, \epsilon = 1.0, W = -1.0$

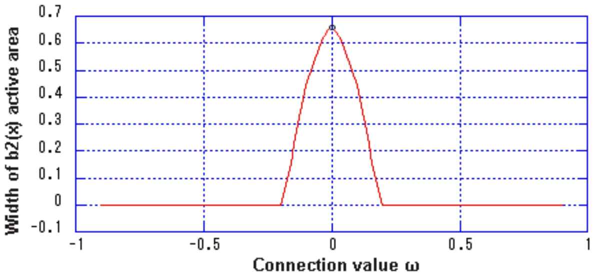

Figure 7: Dependency of width of $b_2(x)$ active area on the parameter ω with $\alpha = 0.0, \beta = 0.49, \epsilon = 1.0, W = -1.0$

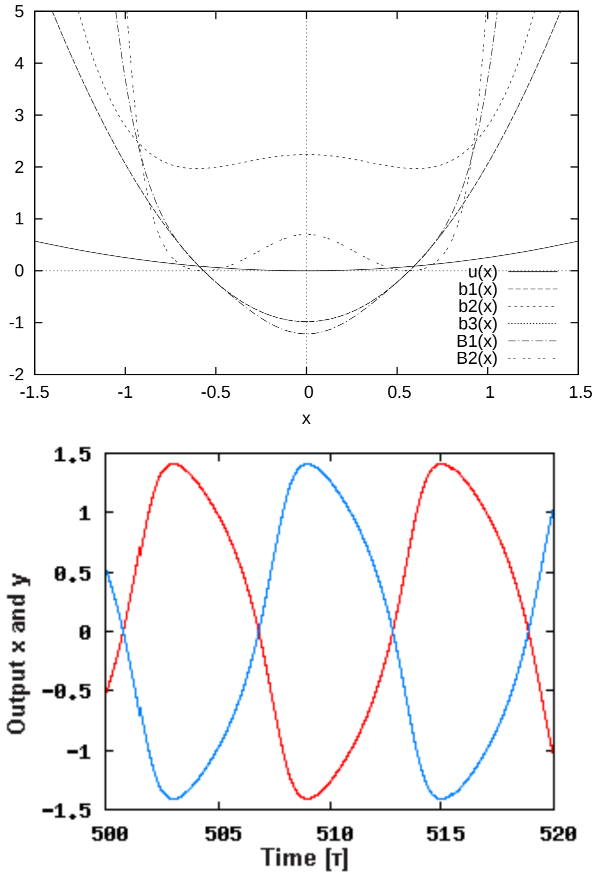

Figure 8: Influence of active areas and b_2 function,when $\alpha = 0.0, \beta = 0.49, \epsilon = 1.0, W = -1.0, \omega = -0.7$.

and hence this function may have much effect on wave forms. When a moving particle gets out from active areas and pass through an extremum of $b_2(x)$ function, the movement becomes slowly(Fig.8). $B_1(x)$ and $B_2(x)$ consist of the product of $b_0(x), b_1(x), b_2(x)$ and $b_3(x)$, consequently these functions have complex forms.

4. Conclusions

In this paper we discussed the universality of the burst dynamics with the concept of a potential function and active areas. This concept is applied to the mutual coupling systems, so we have analyzed parameter dependency of active areas and the relativity of wave forms to the potential function and the active areas, we expect the concept to be helpful for understanding the dynamics of these systems.

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