2011 International Symposium on Nonlinear Theory and its Applications
NOLTA2011, Kobe, Japan, September 4-7, 2011

# Synchronization of a New Hyperchaotic System Using Linear Controllers

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**Abstract**–In this paper three linear controllers are designed for completely synchronizing a new hyperchaotic system. The first one is a linear feedback controller, which uses only one term to synchronize the hyperchaotic system. Then by using Backstepping scheme and Lyapunov stability theory, the second controller is designed, which makes the hyperchaotic system quickly synchronized. The third controller is proposed using direct design method, which needs no deducing the Lyapunov functions in design process. Finally simulation results are presented to demonstrate the effectiveness of the three proposed controllers. The new hyperchaotic system has only one nonlinear term and the proposed controllers are all linear, so they especially fit for secure communication. All the linear controllers proposed in this paper are easy to be realized in actual project applications.

## 1. Introduction

Chaos is very interesting nonlinear phenomenon and has applications in many areas such as biology, economics, signal generator design, secure communication, and so on. Chaotic synchronization has been a hot topic since the pioneering work of Pecora and Carroll [1], variety of methods and techniques have been proposed for controlling chaotic systems.

In recent years, many synchronization methods have been developed, such as OGY control [1], feedback control [2], coupled control [3], adaptive control [4], H∞ control [5] and many others. In these synchronization methods, Backstepping scheme [6] is often applied to analyze the stability of error systems, which reduces the difficulty of analysis. In this paper, Backstepping scheme is also used in theoretical analysis. However most of these synchronization methods are nonlinear controllers, so they are too complicated to be applied in actual projects. In this paper, three linear controllers are designed for a new hyperchaotic system.

On the other hand, a lot of new chaotic systems are found in the last three decades. By reconstructing some famous chaotic systems, researchers obtained some hyperchaotic systems such as hyperchaotic Lorenz system [8], hyperchaotic Chen system [9], hyperchaotic Liu system [10], hyperchaotic Qi system [11] and so on. In ref. [12], Zhou et al found a new four-dimensional hyperchaotic system. This system has only one nonlinear term, which has the same simple structures as hyperchaotic Rössler system [13,14]. The simple structure makes it fit for secure communication.

In this paper, three synchronization controllers are designed for this new hyperchaotic system [12]. The first controller is a linear feedback controller, which has very simply structure with only one term. Using Backstepping scheme and Lypanov stability theory, the second one is designed. It is different from traditional linear feedback method and linear coupled method. Quick synchronization speed can be obtained using this method. And then the third one is obtained using direct design method, which is obtained without deducing Lyapunov functions. Finally simulation results are presented to demonstrate the effectiveness of the three proposed controllers.

## 2. A new hyperchaotic system

In 2009, Zhou P. et al constructed a new four-dimensional hyperchaotic system [12]. The new system is similar to the famous Rössler system. They have only one nonlinear term. The autonomous differential equations are described by:

$$\begin{cases} \dot{x}_1 = ax_1 - 1.2x_2 \\ \dot{x}_2 = x_1 - 0.1x_2x_3^2 \\ \dot{x}_3 = -x_2 - 1.2x_3 - 5x_4 \\ \dot{x}_4 = x_3 + 0.8x_4 \end{cases} \quad (1)$$

The simple structure makes it fit for secure communication and project application. System (1) has three equilibrium points:

$S_0 = (0, 0, 0, 0)$,

$S_{1,2} = \left(\mp 0.12p^3 \pm 5p^3/8, \mp 1.2p \pm 5p/0.8, \pm p, \mp p/0.8\right)$

where $p = \sqrt{12/a}$ .

$\lambda_1 = 0, 0 > \lambda_2 > \lambda_3 > \lambda_4$ and system (1) is in periodic state when $0 < a \le 0.27$ . The Lyapunov exponents are (0, -0.12728, -0.15564, -0.15674) when $a$=0.1.

$\lambda_1 = \lambda_2 = 0, \lambda_3 < 0, \lambda_4 < 0$ and system (1) is in simulant periodic state when $0.27 < a \le 0.45$ . The Lyapunov exponents are (0, 0, -0.02432, -0.4763) when $a$=0.4.

$\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0, \lambda_4 < 0$ and system (1) is in chaotic state when $0.45 < a \le 0.59$ . The Lyapunov exponents are (0.02066, 0, -0.02865, -0.49696) when $a$=0.53.

And when $0.59 < a \le 0.69$ , $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 = 0, \lambda_4 < 0$ and system (1) is in hyperchaotic periodic state. The

Lyapunov exponents are (0.1191, 0.04961, 0, -0.85222) when $a$=0.66.

Fig.1 to Fig.4 show the periodic, simulant periodic, chaotic and hyperchaotic states of system (1).

The complex signals in hyperchaotic systems enhance the safety of chaotic secure communication and chaotic information encryption. So in this paper, system (1) is considered as the drive system when $0.59 < a \le 0.69$.

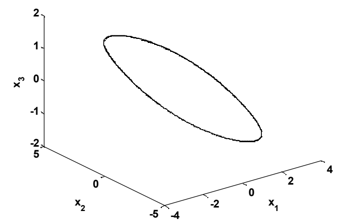

Fig.1 The attractors plot when $a$=0.1

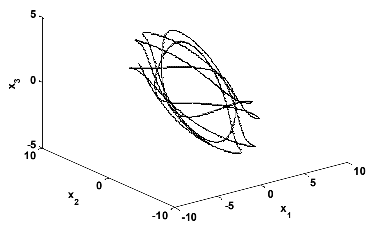

Fig.2 The attractors plot when $a$=0.4

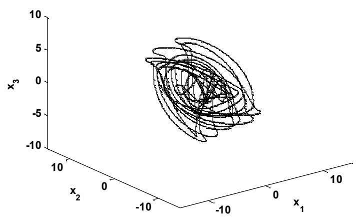

Fig.3 The attractors plot when $a$=0.53

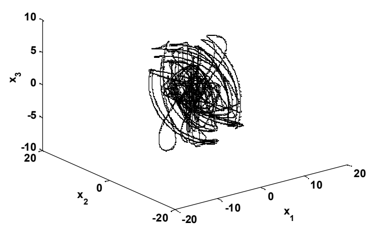

Fig.4 The attractors plot when $a$=0.66

## 3. Linear synchronization methods

### 3.1. Linear feedback control

Linear feedback method is easy to be realized in actual project applications, so this method is often used to control or synchronize chaotic systems [15,16].

Choose the parameter $a$ in system (1) as $0.59 < a \le 0.69$ and the error formats of complete synchronization as $e_i = y_i - x_i \ (i = 1 \sim 4)$. Considering system (1) as the drive system and using traditional linear feedback scheme, the corresponding response system is described by:

$$\begin{cases} \dot{y}_1 = ay_1 - 1.2y_2 + u_1 \\ \dot{y}_2 = y_1 - 0.1y_2y_3^2 + u_2 \\ \dot{y}_3 = -y_2 - 1.2y_3 - 5y_4 + u_3 \\ \dot{y}_4 = y_3 + 0.8y_4 + u_4 \end{cases} \tag{2}$$

where $u_i = -k_ie_i$ and $k_i > 0 (i = 1 \sim 4)$.

Using the similar technique as in Ref. [2], we find that only controller $u_1$ is enough to completely synchronize system (1) and system (2). So the simplified controller is designed as:

$$\begin{cases} u_1 = -k_1(y_1 - x_1) \\ u_2 = u_3 = u_4 = 0 \end{cases} \quad (k_1 > 0) \tag{3}$$

Simulation results in next section will show the effectiveness of controller (3).

### 3.2. Linear method using Backstepping scheme

In this part we will design a new linear synchronization controller for hyperchaotic system (1). This controller is different from normal linear feedback or linear coupled controllers. Backstepping scheme is adopted to design the controller and to analyze the stability of error systems. This method reduces the design difficulty for controlling hyperchaotic systems. This method can be widely used in other hyperchaotic systems.

Also consider system (1) as drive system and systems (2) as response system. So the error systems are:

$$\begin{cases} \dot{e}_1 = ae_1 - 1.2e_2 + u_1 \\ \dot{e}_2 = e_1 - 0.1y_2y_3^2 + 0.1x_2x_3^2 + u_2 \\ \dot{e}_3 = -e_2 - 1.2e_3 - 5e_4 + u_3 \\ \dot{e}_4 = e_3 + 0.8e_4 + u_4 \end{cases} \tag{4}$$

Deeply analyzing the structure of error system (4), there are only two nonlinear terms in $\dot{e}_2$. In order to design a linear controller for error system (4), Backstepping scheme and Lyapunov stability theory are adopted. The deduction sequences are:

$e_1 \to e_2 \to e_3 \to e_4$

If $e_4$ is gradually stable, the other error systems are all gradually stable in turn.

**Step 1:** Set: $w_1 = e_1$, so we get:

$\dot{w}_1 = \dot{e}_1 = ae_1 - 1.2e_2 + u_1$

Given Lyapunov function is $V_1 = w_1^2 / 2$, we get:

$$\dot{V}_1 = w_1\dot{w}_1 = e_1(ae_1 - 1.2e_2 + u_1) \tag{5}$$

If $u_1$ is designed as $u_1 = -k_1e_1$ and $k_1 \ge a$, the formula (5) is obtained as:

$$\dot{V}_1 = -(k_1 - a)e_1^2 - 1.2e_1e_2 \quad (6)$$

So $\alpha_1(w_1)$ is considered as virtual control to $e_2$. When $\alpha_1(w_1) = e_2 = 0$, $\dot{V}_1 = -(k_1 - a)e_1^2 \le 0$. So $e_1$ is gradually stable.

**Step 2:** Set: $w_2 = e_2$, so we get:

$$\dot{w}_2 = \dot{e}_2 = e_1 - 0.1y_2y_3^2 + 0.1x_2x_3^2 + u_2$$

Given Lyapunov function is $V_2 = V_1 + w_2^2/2$, we get:

$$\dot{V}_2 = \dot{V}_1 + w_2\dot{w}_2 = \dot{V}_1 + e_2(e_1 - 0.1y_2y_3^2 + 0.1x_2x_3^2 + u_2) \quad (7)$$

If $u_2$ is designed as $u_2 = -e_1$, the formula (7) is obtained as:

$$\begin{aligned}\dot{V}_2 &= \dot{V}_1 + e_2(e_1 - 0.1y_2y_3^2 + 0.1x_2x_3^2 - e_1) \\ &= \dot{V}_1 + e_2(-0.1y_2y_3^2 + 0.1x_2y_3^2 - 0.1x_2y_3^2 + 0.1x_2x_3^2) \\ &= \dot{V}_1 + e_2[-0.1e_2y_3^2 - 0.1x_2(y_3^2 - x_3^2)] \\ &= \dot{V}_1 + 0.1y_3^2e_2^2 - 0.1x_2(y_3 + x_3)e_2e_3 \end{aligned} \quad (8)$$

So $\alpha_2(w_2)$ is considered as virtual control to $e_3$. When $\alpha_2(w_2) = e_3 = 0$, $\dot{V}_2 = \dot{V}_1 - 0.1y_3^2e_2^2 \le 0$. So $e_1, e_2$ are gradually stable.

**Step 3:** Set: $w_3 = e_3$, so we get:

$$\dot{w}_3 = \dot{e}_3 = -e_2 - 1.2e_3 - 5e_4 + u_3.$$

Given Lyapunov function is $V_3 = V_2 + w_3^2/2$, we get:

$$\dot{V}_3 = \dot{V}_2 + w_3\dot{w}_3 = \dot{V}_2 + e_3(-e_2 - 1.2e_3 - 5e_4 + u_3) \quad (9)$$

If $u_3$ is designed as $u_3 = e_2$, the formula (9) is obtained as:

$$\begin{aligned}\dot{V}_3 &= \dot{V}_2 + e_3(-e_2 - 1.2e_3 - 5e_4 + e_2) \\ &= \dot{V}_2 - 1.2e_3^2 - 5e_3e_4 \end{aligned} \quad (10)$$

So $\alpha_3(w_3)$ is considered as virtual control to $e_4$. When $\alpha_3(w_3) = e_4 = 0$, $\dot{V}_3 = \dot{V}_2 - 1.2e_3^2 \le 0$. So $e_1, e_2$ and $e_3$ are gradually stable.

**Step 4:** Set: $w_4 = e_4$, so we get:

$$\dot{w}_4 = \dot{e}_4 = e_3 + 0.8e_4 + u_4.$$

Given Lyapunov function is $V_4 = V_3 + w_4^2/2$, we get:

$$\dot{V}_4 = \dot{V}_3 + w_4\dot{w}_4 = \dot{V}_3 + e_4(e_3 + 0.8e_4 + u_4) \quad (11)$$

If $u_4$ is designed as $u_4 = -e_3 - k_4e_4$ and $k_4 \ge 0.8$, the formula (11) is obtained as:

$$\begin{aligned}\dot{V}_4 &= \dot{V}_3 + e_4(e_3 + 0.8e_4 - e_3 - k_4e_4) \\ &= \dot{V}_3 - (k_4 - 0.8)e_4^2 \le 0 \end{aligned} \quad (12)$$

So from formula (6), (8), (10) and (12), it is easy to find that $e_i$ $(i = 1 \sim 4)$ are all gradually stable.

That is to say, when linear control is designed as:

$$\begin{cases} u_1 = -k_1e_1 \\ u_2 = -e_1 \\ u_3 = e_2 \\ u_4 = -e_3 - k_4e_4 \end{cases} \quad (k_1 \ge a, k_4 \ge 0.8) \quad (13)$$

system (1) and system (2) will achieve complete synchronization.

Linear controller (13) owns simple structure, so it is very suitable to be applied in actual projects.

### 3.3. Linear method using direct design method

In last part we design another linear controller using Backstepping scheme and Lyapunov stability theory. In this part the third linear controller will be proposed using the direct design method without deducing Lyapunov functions.

To error system (4), system (1) and system (2) will be completely synchronized if $\lim\limits_{t\to\infty} e_i = 0$.

Error system (4) can be constructed as the following:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = \begin{bmatrix} a & -1.2 & 0 & 0 \\ 1 & -y_3^2 & -x_2(y_3 + x_3) & 0 \\ 0 & -1 & -1.2 & -5 \\ 0 & 0 & 1 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \quad (14)$$

The linear controller is designed as:

$$\begin{cases} u_1 = -k_1e_1 \\ u_2 = -e_1 - k_2e_2 \\ u_3 = e_2 \\ u_4 = 0 \end{cases} (k_1 > a, k_2 > 0) \quad (15)$$

Then the error system is constructed as:

$$\begin{aligned}\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} &= \begin{bmatrix} a - k_1 & -1.2 & 0 & 0 \\ 0 & -k_2 - y_3^2 & -x_2(y_3 + x_3) & 0 \\ 0 & 0 & -1.2 & -5 \\ 0 & 0 & 1 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \\ &= \begin{bmatrix} a - k_1 & -1.2 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \\ 0 & 0 & -1.2 & -5 \\ 0 & 0 & 1 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -y_3^2 & -x_2(y_3 + x_3) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \\ &:= A(t)\begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix}^T + O(e,t) \end{aligned} \quad (16)$$

where $A(t)$ is linear system matrix, $O(e,t)$ is nonlinear term.

According to the the stability theory of linear system and lemma in Ref. [17], $O(0,t) = 0$ and $\lim\limits_{\|x\|\to 0} \frac{\|O(x,t)\|}{\|x\|} = 0$ to all $t$. So the error system (16) is gradually stable if all the real parts of eigenvalues in matrix $A(t)$ are negative.

$$|\lambda I - A(t)| = \left\| \begin{bmatrix} \lambda - a + k_1 & 1.2 & 0 & 0 \\ 0 & \lambda + k_2 & 0 & 0 \\ 0 & 0 & \lambda + 1.2 & 5 \\ 0 & 0 & -1 & \lambda - 0.8 \end{bmatrix} \right\|$$

$$= (\lambda - a + k_1)(\lambda + k_2)(\lambda^2 + 0.4\lambda + 4.04) \quad (17)$$

From formula (17), we get:

$$\lambda_1 = a - k_1, \lambda_2 = -k_2, \lambda_{3,4} = -0.2 \pm 2i \quad (18)$$

Obviously when $k_1 > a$ and $k_2 > 0$, all the real parts of eigenvalues in matrix $A'(e)$ are negative. So system (1) and system (2) will receive complete synchronization under controller (15).

## 4. Simulation researches

Firstly, considering hyperchaotic system (1) as the drive system and system (2) as the response system, the controller (3) is used in simulations. The system parameters are set as $a = 0.66, k_1 = 5$ and the initial values of two systems are set as (1,2,2,1) and (30,-40,50,-60).

In simulations, the step value is 0.05. We plot the curves of complete synchronization error. Fig.5 is the chaotic attractors of the two chaotic systems. Fig.6 is the plot curves of the error systems. The two plots show that the complete synchronization is received between system (1) and (2). In about 20 seconds, all the error systems are stable to 0.

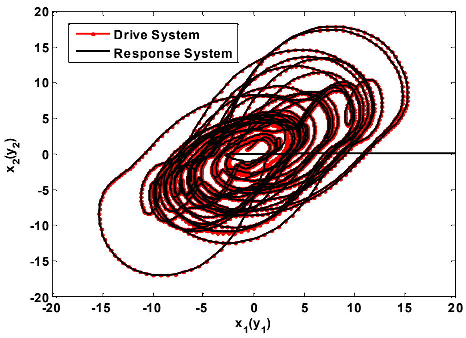


Fig. 5 The attractors of two systems under the linear controller (3)

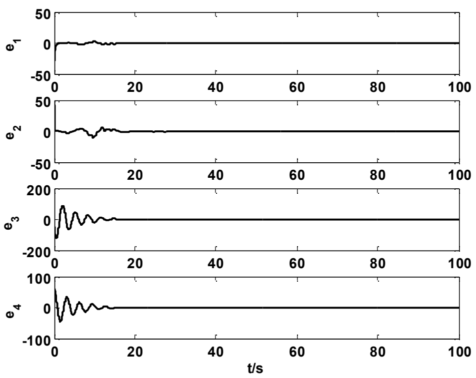


Fig. 6 The error plot under the linear controller (3)

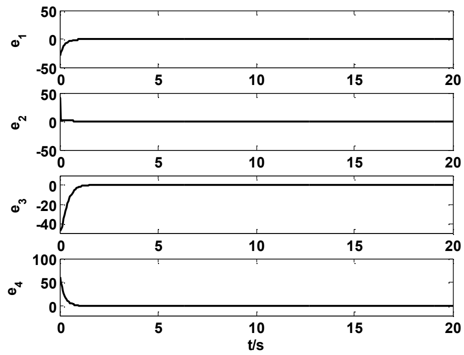


Fig. 7 The error plot under the linear controller (13)

Secondly, controller (13) is used to synchronize system (1) and system (2). $k_1 = k_4 = 5$ and other parameters are same with the first simulation. Fig. 7 is the plot curves of the error systems. Compare fig.6 with fig.7, controller (13) quickens the synchronization speed and all the two systems are synchronized in about 2 seconds.

Lastly, controller (15) is used to synchronize system (1) and system (2). $k_1 = 5$ and other parameters are same with the first simulation. Fig. 8 is the plot curves of the error systems. All the two systems are synchronized in about 13 seconds.

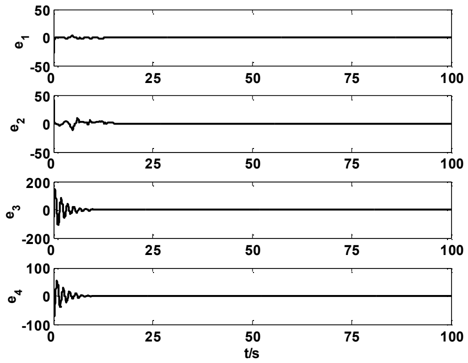


Fig. 8 The error plot under the linear controller (15)

From the three simulation results, they obviously show that the controller (13) has the fastest synchronization speed for it adds control terms in every sub systems.

## 5. Conclusions

In this paper, synchronization control methods of a new hyperchaotic system are proposed. The new hyperchaotic system has only one nonlinear term, so it fits for secure communication. The first controller is based on linear feedback method, which has the simplest structure. The second controller is designed using Backstepping scheme and Lyaounov stability theory, which obtains the fastest synchronization speed. The third controller is proposed using direct design method without deducing Lyapunov functions, which simplifies the academic analysis of error systems. The simulations show the hyperchaotic systems are completely synchronized via the three methods.

The idea in this paper can be used in other hyperchaotic systems, which is different from traditional feedback methods and coupled control methods. Our Future works regarding this topic include the investigation of some other types of synchronization for this class of hyperchaotic systems, including some uncertain parameters.


**Acknowledgments**

This work was supported by China Postdoctor Science Foundation of No.20080430170, Jiangsu Postdoctor Science Foundation of No. 0801046B.


## References


[1] L. Pecora, T. Carroll. "Synchronization in chaotic systems," *Physical Review Letters*. vol. 64, no. 8, pp. 821–825, Feb. 1990.

[2] G. Gambino, M. C. Lombardo, M. Sammartino. "Global linear feedback control for the generalized Lorenz system," Chaos, Solitons and Fractals. vol. 29, no. 2, pp. 829–837, Feb. 2006.

[3] F.X. Chen, C.S. Zhang, G.J. Ji, et al. "Lu system chaos synchronization using single variable feedback," *Proceedings of 29h Chinese Control Conference,* China, July 29-31 2010, pp: 474-477,

[4] H.H. Chen, C.I. Lee, P.H. Yang. "Adaptive synchronization of linearly coupled unified chaotic systems," *Chaos, Solitons and Fractals.* vol. 40, no. 2, pp. 598–606, Apr. 2009.

[5] Y.F. Feng, Q.L. Zhang. "Adaptive synchronization of uncertain chaotic systems via switching mechanism," *Chinese Physics*, vol. 19, no.12, pp: 100504-7, Dec. 2010.

[6] H. Huang, G. Feng. "Robust H∞ synchronization of chaotic Lur'e systems," *Chaos*, vol. 36, no. 3, pp: 3113-3122, Mar. 2008

[7] N.P. Hossein, N.Jalili, H.M. Seyed, et al. "Robust adaptive backstepping control of uncertain Lorenz system," *Chao*, vol.20, no.2, pp: 3105–3109, June 2010

[8] G.Y. Wang, Y. Zheng, J.B. Liu. "A hyperchaotic Lorenz attractor and its circuit implementation," *Acta Physica Sinica*, vol. 56, no.6, pp: 3113-3120, June 2007. (in Chinese)

[9] Y.X. Li, W.K. Tang, G.R. Chen. "Generating hyperchaos via state feedback control," *Int. Journal of Bifurcation and Chaos*, vol. 15, no.10, pp: 3367-3375, Oct. 2005.

[10] F.Q. Wang, C.X. Liu. "Hyperchaos evolved from the Liu chaotic system," *Chinese Physics*, vol. 15, no.5, pp: 963-968, May 2006.

[11] G. Qi, M.A. Wyk, B.J. Wyk. "A new hyperchaotic system and its circuit implementation, vol.40, no. 5, pp: 2544-2549, June. 2009

[12] Ping Zhou, L.J. Wei, X.F. Chen. "A hyperchaotic system with only one nonlinear term," *Acta Physica Sinica*, vol. 58, no. 8, pp: 5201-5208, Aug. 2009 (In Chinese)

[13] O.E. Rössler. "An equation for hyperchaos," *Pyhsics Letters A*, vol. 71, no. 2, pp: 397-398, Apr. 1979

[14] Q.D. Li. "A topological horseshoe in the hyperchaotic Rossler attractor," *Pyhsics Letters A*, vol. 372, no. 17, pp: 2989-2994, Apr. 2008

[15] H.H. Chen. "Stability criterion for synchronization of chaotic systems using linear feedback control," *Pyhsics Letters A*, vol. 372, no. 11, pp: 1841-1850, Mar. 2008

[16] G.L. Cai, S. Zheng, L.X. Tian. "A new hyperchaotic system and its linear feedback control," *Chinese Physics B*, vol. 17, no. 11, pp: 4039-4046, Nov. 2008

[17] J.H. Lü, D. Chen. "A new chaotic system and beyond: the generalized Lorenz-like system," *Int. J. Bifurcat. Chaos*, vol. 14, no. 5, pp: 1507-1537, May 2004