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A Structual Detection of Unstable Periodic Points in Chaotic Attractors by the Directional Coloring

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Abstract

We propose an application of the visualization method called directional coloring by using mapping direction of the difference equation. An invariant pattern can be detected for a given chaotic attractor with this coloring. Unstable periodic points(UPPs) are visualized as a crossing points of different colors and these locations are determined by the pattern. With this property and thanks to Newton's mehtod, we can compute UPPs systematically and accurately. We show numerical results and their fractal nature.

1. Introduction

If we obtain a good mathematical model to describe dynamical behavior of the target system with appropriate ways [2], we can extract much information about the nonlinear phenomena from the model, such as local and global bifurcations of singular points[4], chaos[7], and synchronization of oscillations[5], and so on. While, even for a chaos observed in a low-dimensional dynamical system, many problems still remain unsolved[6]. For example, computation of locations of unstable periodic orbits embedded in the chaos, identification of saddles causing crises, relationship among manifolds of saddles and chaos are still challenging problems.

In this paper, we consider featuring direction information between a current point and its n-time mapped point into coloring, i.e., an argument defined by these two points is utilized for coloring. This can depict not only orientation or tendency of an orbit within the chaotic attractor but also vector field of the system. For some numbers for n, we have unique patterns reflected from own nonlinearity. By this method, embedded UPPs in a chaos attractor are visualized clearly by concentrating points of colors. Although it is difficult to distinguish the locations of UPPs embedded in chaotic attractors visually by the conventional methods, but the proposed method can specify them as distinguishable points. Finally, as an application, we propose an UPP detector by using the directional coloring results.

For one dimensional discrete systems $x_{k+1} = f(x_k)$, we visually tell where the periodic points are, indeed, the cross points of the graph of f^n and $x_{k+1} = x_k$ indicates them. Our method gives similar intuitive information about UPPs.

2. Description in question and the visualization technique

The following two-dimensional nonlinear difference equation is considered:

$$\begin{aligned} x_{k+1} &= f(x_k, y_k, \lambda) \\ y_{k+1} &= g(x_k, y_k, \lambda) \end{aligned} \tag{1}$$

where x_k and y_k are state variables, λ is a parameter, f and g are C^∞-class functions. With vector expressions $\boldsymbol{x} = (x, y)$, and $\boldsymbol{f} = (f, g)$, the formula (1) is expressed as follows:

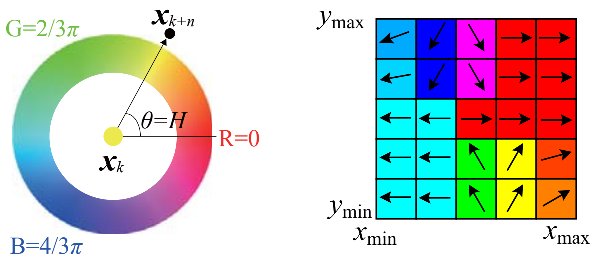
$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k, \lambda). \tag{2}$$

Then n-times iterated map $\boldsymbol{x}_{k+n}$ for $\boldsymbol{x}_k$ is given as

$$\boldsymbol{x}_{k+n} = \boldsymbol{f}^n(\boldsymbol{x}_k, \lambda) \tag{3}$$

The directional coloring method[1] is very easy, that is, the current point is colored by the angle θ for the n iterated point, see Fig.1(a). We only use the hue information. Both saturation and intensity are fixed through this work. Fig.1(b) explain an example coloring. It is noteworthy that this coloring process does not require a high performance computation. To obtain basins of attractions, many iterations and finding periodicity are needed. The results of them look similar, but the directional coloring can be obtained many times quicker.

In Fig.1(b), arrows show the average mapping directions in each pixel (corresponding to a certain domain). A geographical properties of $\boldsymbol{f}^n$ are reflected to the the coloring result, that is, color changing looks smooth since $\boldsymbol{f}^n$ is C^∞. We can assume changing of θ is smooth about the UPP. Using this assumption, we can detect 2-dimensionally UPPs (repellers) with template matching method. It works successful, but



(a)Directional Coloring (b)State Plane

Figure 1: Visualization method

it fails for saddle points. The reason of this failure is related with the shape of basin boundaries. For example, in a chaotic attractor for the following chaotic map:

$$\begin{cases} x_{k+1} &= y_k + a x_k \\ y_{k+1} &= x_k^2 + b \end{cases} \quad (4)$$

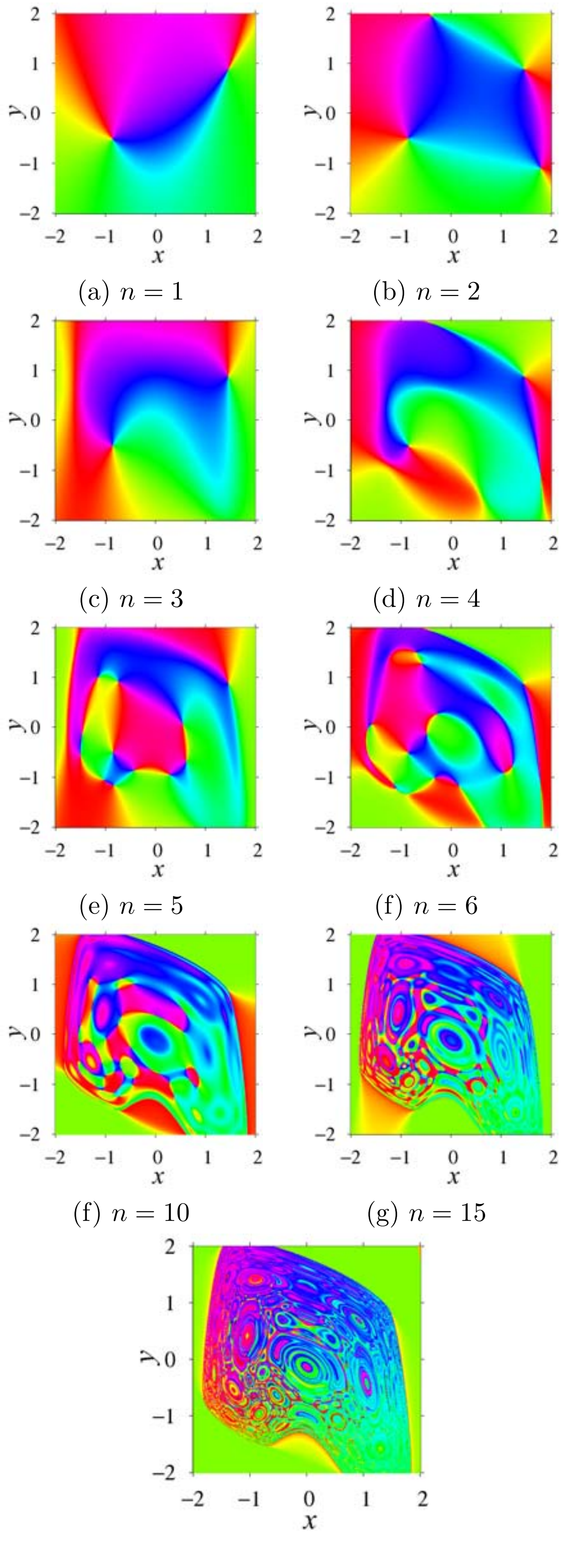
where $a = 0.4$, $b = -1.24$, stable manifolds construct a millefeuille structure in the state plane. The flows inside a basin tend to keep a certain direction along the stable manifolds, thus the θ changes sharply around the saddle.

The directional coloring results are shown in Fig.2(a)–(h). Since there is a chaotic attractor in this parameter value, a basin boundary for stable periodic points is not defined, however, they look like basin boundary pictures which can be seen in other parameter values. It turns out that change of a color is concentrating in a certain point of seeing these pictures. We investigated the map direction from the surrounding color of this point. As a result, it turned out that the point which is concentrating the color is an UPP.

3. Unstable Periodic Point Detector

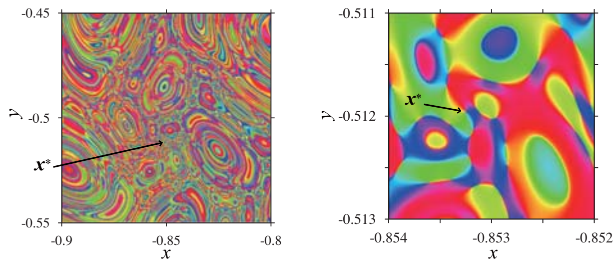
Theoretically, any periodic point is unstable inside a chaos attractor, i.e., any periodic point is a saddle, unstable node, or unstable focus. Detecting UPPs has been studied for a long time, and many efficient methods have been proposed.

As mentioned above, these periodic orbits are expressed visibly as concentrating points in the directional coloring. As an application of the directional coloring, we propose a simple method which can detect n-periodic UPPs in a given attractor. Firstly we find some candidate pixels S whose eight surrounding pixels make the hue circle centering on S. This scheme is a simple image processing, there are no technical difficulties. Although this method also detect periodic points whose period is a divisor of n, a good initial guess can be supplied if the user narrows the searching area for S. But Next we give this candidate points



(a) $n = 1$ (b) $n = 2$ (c) $n = 3$ (d) $n = 4$ (e) $n = 5$ (f) $n = 6$ (f) $n = 10$ (g) $n = 15$ (h) $n = 20$

Figure 2: The directional coloring for Eq.(4) in x-y plane. $a = 0.4$, $b = -1.24$



(a) (b)

Figure 3: Directional coloring for Eq.(4) with $n = 37$. (a): around the fixed point, (b): an magnification of (a).

to Newton's method. The condition of the periodic points are given in Eq.(5).

$$\boldsymbol{f}^n(\boldsymbol{x}) - \boldsymbol{x} = 0 \quad (5)$$

The Jacobian matrix is obtained by solving the variational equation of Eq.(2). Newton's method ensures improve accuracy of locations of the UPPs from the candidates. Fig.3(a) shows a complicated structure by the directional coloring for $n = 37$ around the fixed point $\boldsymbol{x}^*$, but by scaling this area, a simple structure is obtained, see, Fig. 3(b). One can easily confirm the positions of periodic points visually. The first-guess supplier also can detect candidate points accurately. In the case $n = 79$, the initial guess obtained from the image is $(-0.85325626090, -0.51195375578)$, indicated as $\boldsymbol{c}$ in Fig.4. With three times iteration of Newton's method, an accurate location of 79-periodic point $(-0.85325626092671, -0.51195375577927)$ is obtained with a 10^{-15} error. By iterating Eq.(4) with this solution, we have the other 78 periodic points, and they appear as visible concentrating points in this figure except for $\boldsymbol{x}^*$. They are unstable nodes since the multipliers are 3.4×10^8 and -1.4×10^9. Furthermore, they are proper 79-periodic points since n is a prime number. Note that the repellers in this system cannot be obtained by inverse time simulation since this map is not invertible.

4. Invariant pattern and its fractal nature

One can notice that coloring patterns of Fig.3(a) and Fig.4 are similar, in fact, if one of them is rotated π radian about $\boldsymbol{x}^*$, they are almost coincident. We experimentally confirmed that there definitely exists this invariant minimum pattern around $\boldsymbol{x}^*$ for any n under an allowance of its rotation. The color assignment of the pattern is also invariant and periodic points located in this pattern can be enumerated. The position of a periodic point $\boldsymbol{c}$ defined above is actually the nearest n-periodic point and it is relatively invariant from

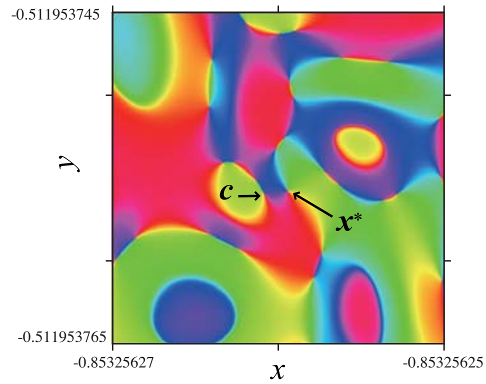


Figure 4: Directional coloring for Eq.(4) with $n = 79$. This figure show an invariant minimum pattern. Many UPPs are visualized.

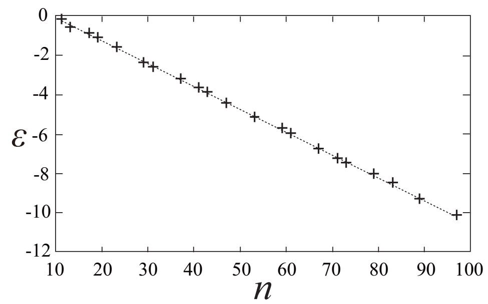


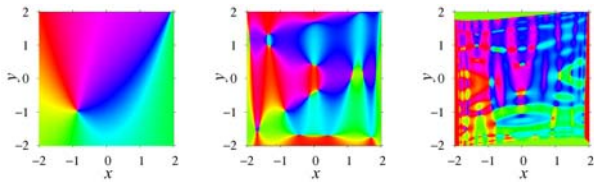
Figure 5: A scale property of the invariant minimum patterns. We choose prime numbers for n, $11 \leq n \leq 97$. The fitted straight line is $\varepsilon \approx -0.116n + 1.076$.

$\boldsymbol{x}^*$. We utilize this point for measuring the scale. Let ε be a logarithm of an Euclid distance between $\boldsymbol{c}$ and $\boldsymbol{x}^*$. We compute ε to measure the scale of each invariant minimum pattern.

Fig.5 show clearly a scale law, also demonstrate that the detection of periodic points with the directional coloring is worked out quite accurately. For every step for increment of n, the invariant pattern shrinks to 58.6 percent for each, and its rotated with a certain degree, and it it embedded into the state space. As shown in Figs.3(b) and 4, the number of periodic points appeared in the invariant minimum pattern is fixed, therefore a finite number of nesting for the invariant minimum pattern determines the total number n. A fractal structure observed in a chaos attractor usually refers Julia sets, i.e., infinite times magnification of the set cannot reach a simple structure. For coloring results for Eq.(4), a finite number n means a finite order of nesting, therefore this enables us to detect UPPs with a specified number of period. This property is possibly special for Eq.(4), however, it is notable that the directional coloring can point out it.

5. In the Case of Another Parameter

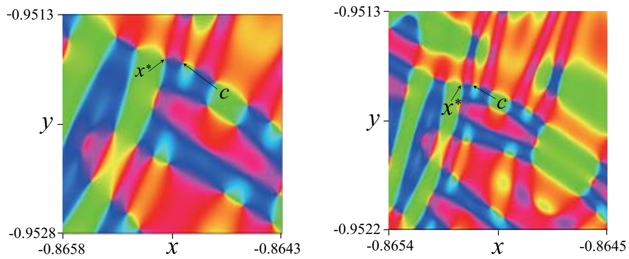
Next, we analyze in the case of another parameter value. Let us assume $a = -0.1$ and $b = -1.7$. Also chaotic attractor is given with this parameter.



(a) $n = 1$ (b) $n = 5$ (c) $n = 10$

Figure 6: The directional coloring for Eq.(4) in x-y plane. $a = -0.1$, $b = -1.7$.

Fig.6(a)–(c) show coloring results by using directional coloring with $n = 1, 5, \text{and} 10$, respectively. The common fixed point shared in all figures is $\boldsymbol{x}^* = (-0.865097186201, -0.951606906110)$. And it is turned out that $\boldsymbol{x}^*$ =repeller.



(a) $n = 36$ (b) $n = 40$

Figure 7: Directional coloring for Eq.(4) with $n = 36, \quad 40$. This figure show an invariant minimum pattern. Many UPPs are visualized.

Fig.7(a) show complicated structures visualized by the directional coloring for $n = 36$ and 40 around the fixed point $\boldsymbol{x}^*$. There is a pattern relating the generation of periodic points for given n, that is, we can also find a new created cross point near $\boldsymbol{x}^*$ as $\boldsymbol{c}$ in both figures.

Fig.8 show clearly a scale law. This result also demonstrate the fractal nature. If one locate $\boldsymbol{c}$ once with arbitrary number of k with Newton's method, any location of $\boldsymbol{c}$ is computed (predicted) by the the scaling fact ε for any number of n. Note that other $n-1$ points are given by repeating Eq.(2) simply. Then we have a systematical computation of UPPs by the directional coloring.

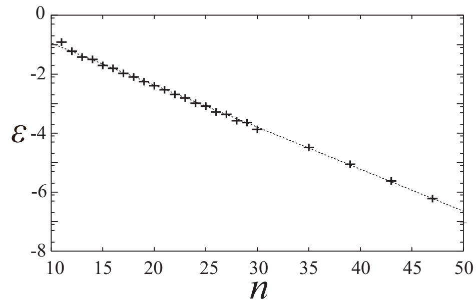


Figure 8: A scale property of the invariance in the pattern. The fitted straight line is $\varepsilon \approx -0.1202n + 0.4441$.

6. Conclusion

The directional coloring reveals an invariant pattern hidden in the chaotic attractor. With this method, UPPs with the specific number of period is visualized. As an application, a detection scheme of these UPPs is shown. As the future works, reason for organization of the invariant patterns should be analyzed[3].

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