ICTF 2016 Keynote Speech

# Oscillation Model for Describing Propagation of Activities on Network Caused by Asymmetric Node Interactions

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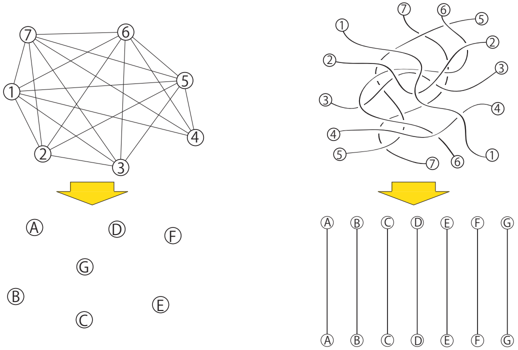
# Background & Research Objective

- Information exchange on social NWs is being activated by the popularity of information NWs.
  $\Rightarrow$ Complex dynamics for describing propagation of activities on the social and information NWs is a rich source of research topics.
- The interaction between nodes (users) is asymmetric (it depends on the direction of links), in general.
- Link asymmetry is described by a directed graphs, and the directed graph is normally expressed by an asymmetric matrix.
- On the other hand, symmetric matrix-based model for network dynamics is mathematically tractable.
- Some types of asymmetric interaction can be analyzed using a symmetric matrix-based model.

Objective: Analysis of oscillation dynamics on NWs caused by asymmetric node interactions

# Advantages of Symmetric Matrix-Based Model

- Any real symmetric matrices can be diagonalized.
- The meanings of diagonalization: To simplify the relation between nodes.




- Advantage: Network dynamics can be analyzed by decomposition into independent simple models.

# Representation of NW structure by Matrix

- Let $\mathcal{G} = \mathcal{G}(V, E)$ be a directed graph with $n$ nodes $\{1, 2, \ldots, n\}$ (where $V$ is the set of nodes and $E$ is the set of links).
- Let $w_{ij} > 0$ be the link weight of the directed link $i \to j$ of $\mathcal{G}$.
- The adjacency matrix $\mathcal{A} = [\mathcal{A}_{ij}]$ is defined as follows:

$$\mathcal{A}_{ij} := \begin{cases} w_{ij}, & \text{(if directed link } (i \to j) \in E) \\ 0, & \text{(if directed link } (i \to j) \notin E) \end{cases}$$

- The adjacency matrix $\mathcal{A}$ for directed graph is asymmetric, in general.
- The adjacency matrix $\mathcal{A}$ can be applied to investigate the structure of $\mathcal{G}$ algebraically.
  ex) If $w_{ij} = 1$ for all the links, $(i, j)$ component of $\mathcal{A}^k$ describes the number of paths from node $i$ to $j$ with length of $k$.

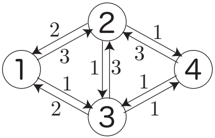
# Definition of Laplacian Matrix

- The weighted out-degree $d_i$ of node $i$ $(i = 1, 2, \ldots, n)$ is defined as

$$d_i := \sum_{j \in \partial i} w_{ij}.$$

- The degree matrix is defined as $\mathcal{D} := \mathrm{diag}(d_1, d_2, \ldots, d_n)$.
- The Laplacian matrix $\mathcal{L}$ is defined as

$$\mathcal{L} := \mathcal{D} - \mathcal{A}.$$

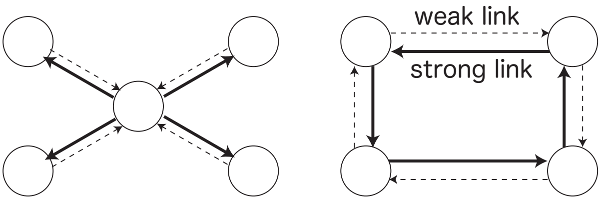



$$\mathcal{D} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad \mathcal{A} = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix} \quad \mathcal{L} = \begin{bmatrix} 4 & -3 & -1 & 0 \\ -2 & 4 & -1 & -1 \\ -2 & -3 & 6 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix}$$

- The Laplacian matrix $\mathcal{L}$ for directed graphs is asymmetric, in general.

# Classification of Link Asymmetry

- Examples of typical asymmetric links:




(a) hub type relation (b) cyclic relation

- In (a), link asymmetry can be reduced to a node characteristic.
  $\Rightarrow$ Asymmetric links can be symmetrized.
- In (b), link asymmetry is purely link's characteristic. Asymmetric links cannot be symmetrized.
- In actual networks, type (a) relations are frequently appeared.
  ex) A major blogger and his/her followers

# Symmetrization of Laplacian Matrix (1)

- $\mathcal{L}$ has left eigenvector ${}^t\boldsymbol{m}$ associated with eigenvalue 0, as

$$ {}^t\boldsymbol{m}\,\mathcal{L} = 0. $$

We assume that all components $m_i > 0$ of ${}^t\boldsymbol{m} = (m_1, \ldots, m_n)$ satisfies

$$ m_i\, w_{ij} = m_j\, w_{ji} \; (\equiv k_{ij}). $$

The physical meaning of this condition is discussed later.

- Let $G = G(V, E)$ be the undirected graph whose link weight is $k_{ij}$ $(= k_{ji})$, and we define the Laplacian matrix $L$ for $G$ as follows:

$$ L := D - A $$

$$ \text{where} \quad A_{ij} := \begin{cases} k_{ij}, & ((i,j) \in E), \\ 0, & ((i,j) \notin E), \end{cases} $$

$$ D := \mathrm{diag}\left( \sum_{j=1}^{n} k_{1j}, \sum_{j=1}^{n} k_{2j}, \ldots, \sum_{j=1}^{n} k_{nj} \right). $$

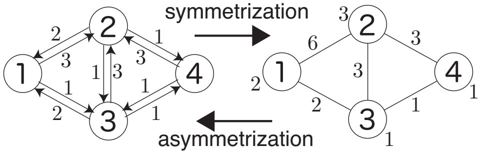
# Symmetrization of Laplacian Matrix (2)

- The original Laplacian matrix $\mathcal{L}$ for the directed graph can be expressed by decomposition as

$$\mathcal{L} = M^{-1}\,L.$$

where $M = \mathrm{diag}(m_1, m_2, \ldots, m_n)$.

$$\mathcal{L} = \begin{bmatrix} 4 & -3 & -1 & 0 \\ -2 & 4 & -1 & -1 \\ -2 & -3 & 6 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix}$$



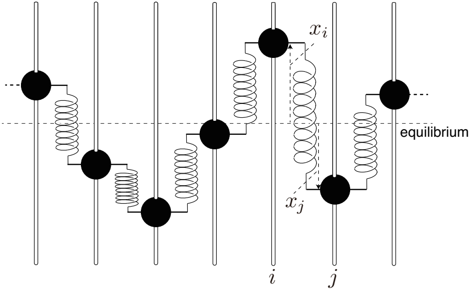

$$= \begin{bmatrix} 4 & -6/2 & -2/2 & 0 \\ -6/3 & 4 & -3/3 & -3/3 \\ -2/1 & -3/1 & 6 & -1/1 \\ 0 & -3/1 & -1/1 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & -6 & -2 & 0 \\ -6 & 12 & -3 & -3 \\ -2 & -3 & 6 & -1 \\ 0 & -3 & -1 & 4 \end{bmatrix}$$

- We define the scaled Laplacian matrix $S$ ($S$ is a symmetric) as,

$$S := M^{-1/2}\,L\,M^{-1/2}.$$

# Oscillation Phenomena on Networks (1)

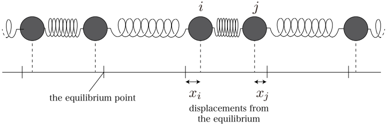
- To describe the propagation of activity of a node through networks, let us consider oscillation dynamics on networks.



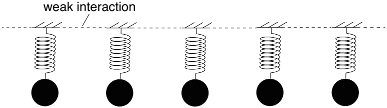

- Let weight $x_i$ of node $i$ be displacement from the equilibrium, and let its restoring force be proportional to the difference in the displacements of adjacent nodes.
- Although the figure shows a 1-dimensional network, it is easily extended to general networks.

# Oscillation Phenomena on Networks (2)

- The oscillating model can be described by longitudinal wave only if we consider 1-dimensional network.




- Difference from well-known Kuramoto model:




  - The same (or similar) oscillators are weakly coupled each other through NWs.
  - For the phase $\theta_i(t)$ of oscillator $i$, natural frequency $\omega$, and a constant $\epsilon$,

$$\frac{\mathrm{d}\theta_i(t)}{\mathrm{d}t} = \omega + \epsilon \sum_{j \in \partial i} \sin(\theta_j(t) - \theta_i(t)).$$

# Oscillation Phenomena on Networks (3)

- Universality of our oscillation model:
  - Consider a general influence force between adjacent nodes $f$.
    - $f = f(\Delta x)$ is a function of the difference of adjacent node states $\Delta x := x_i - x_j$.
    - There is no influence between nodes if $\Delta x = 0$, that is $f(0) = 0$.
    - If $\Delta x = 0$ for all adjacent nodes, the network is in the ground state (equilibrium).
  - By applying Taylor expansion to $f(\Delta x)$ around $\Delta x = 0$,

$$f(\Delta x) = -k_{ij}\,\Delta x + O(\Delta x^2).$$

  - The lowest degree term is $O(\Delta x)$.
  - $f$ is a restoring force, so $k_{ij} > 0$.
  - $O(\Delta x^2)$ describes non-linear effects of the restoring force $f$.
  - For small $\Delta x$, our oscillation model can be applied to various types of $f$.

# Oscillation Model on Networks (1)

- We assign the spring constant for each link $i$–$j$ as the link weight $k_{ij} > 0$.
- We assign the mass of node $m_i > 0$ for each node $i$.
- The Hamiltonian $\mathcal{H}$ of our coupled oscillators system:

$$\mathcal{H} := \sum_{i \in V} \frac{(p_i)^2}{2\, m_i} + \sum_{(i,j) \in E} \frac{k_{ij}}{2} \, (x_i - x_j)^2$$
$$= \sum_{i \in V} \frac{(p_i)^2}{2\, m_i} + \frac{1}{2} \, ({}^t\boldsymbol{x} \, L \, \boldsymbol{x}),$$

  where $p_i$ denotes the conjugate momentum of $x_i$.
  $\boldsymbol{x} = {}^t(x_1, \ldots, x_n)$.
- The canonical equations of motion:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial x_i} = -\sum_{j \in V} L_{ij} \, x_j, \qquad \frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial p_i} = \frac{p_i}{m_i}.$$

# Oscillation Model on Networks (2)

- The EoM of the displacement vector $\boldsymbol{x} = {}^t(x_1, \ldots, x_n)$:

$$M \frac{\mathrm{d}^2\boldsymbol{x}(t)}{\mathrm{d}t^2} = -L\,\boldsymbol{x}(t).$$

- By multiplying $M^{-1}$ from the left, EoM can be expressed using the asymmetric Laplacian matrix $\mathcal{L}$, as

$$\frac{\mathrm{d}^2\boldsymbol{x}(t)}{\mathrm{d}t^2} = -M^{-1}\,L\,\boldsymbol{x}(t) = -\mathcal{L}\,\boldsymbol{x}(t). \tag{1}$$

  The link asymmetry can be represented as the mass of node.

- The condition $m_i\,w_{ij} = m_j\,w_{ji}$ (for symmetrization of $\mathcal{L}$) corresponds to Newton's 3rd law about an action and its reaction.
- By using $y_i = \sqrt{m_i}\,x_i$, $\boldsymbol{y} = {}^t(y_1, \ldots, y_n)$, EoM can be expressed as

$$\frac{\mathrm{d}^2\boldsymbol{y}(t)}{\mathrm{d}t^2} = -S\,\boldsymbol{y}(t). \tag{2}$$

  Oscillation dynamics is described by the symmetric matrix $S$.

# Eigenvalues & Eigenvectors of $S$

- The quadratic form of the scaled Laplacian matrix $S$:

$$ {}^t\boldsymbol{y}\, S\, \boldsymbol{y} = \sum_{(i,j)\in E} k_{ij} \left( \frac{y_i}{m_i} - \frac{y_j}{\sqrt{m_i m_j}} \right)^2 \geq 0. $$

  So, all the eigenvalue of $S$ are non-negative and minimum value is 0.
- Numbering the eigenvalues of $S$ in ascending order of their values:

$$ 0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} $$

- The eigenvector $\boldsymbol{v}_\mu$ associated with the eigenvalue $\lambda_\mu$ $(\mu = 0,\, 1,\, \ldots,\, n-1)$ can be chosen as the eigenbasis:

$$ S\, \boldsymbol{v}_\mu = \lambda_\mu\, \boldsymbol{v}_\mu, \quad \boldsymbol{v}_\mu \cdot \boldsymbol{v}_\nu = \delta_{\mu\nu}, $$

  where $\delta_{\mu\nu}$ denotes the Kronecker delta.

# Solutions of Oscillation Dynamics on NWs (1)

- By substituting $\boldsymbol{y}(t)$ into EoM (2) after expanding it by the eigenbasis $\boldsymbol{y}(t) = \sum_{\mu=0}^{n-1} a_\mu(t)\, \boldsymbol{v}_\mu$,

$$\sum_{\mu=0}^{n-1} \frac{\mathrm{d}^2\, a_\mu(t)}{\mathrm{d}t^2}\, \boldsymbol{v}_\mu = -\sum_{\mu=0}^{n-1} \lambda_\mu\, a_\mu(t)\, \boldsymbol{v}_\mu.$$

- By extracting EoM for each mode,

$$\frac{\mathrm{d}^2\, a_\mu(t)}{\mathrm{d}t^2} = -\lambda_\mu\, a_\mu(t).$$

- Solution of the EoM for each mode is

$$a_\mu(t) = c_\mu\, e^{\pm \mathrm{i}\, (\omega_\mu\, t + \theta_\mu)}, \tag{3}$$

where $\omega_\mu = \sqrt{\lambda_\mu}$, $\mathrm{i} = \sqrt{-1}$, $\theta_\mu =$ phase, $c_\mu =$ constant.

# Solutions of Oscillation Dynamics on NWs (2)

- The solution of EoM (2):

$$\boldsymbol{y}(t) = \sum_{\mu=0}^{n-1} c_\mu \, e^{\pm \mathrm{i}\,(\omega_\mu \, t + \theta_\mu)} \, \boldsymbol{v}_\mu. \tag{4}$$

- The solution of the original EoM (1):

$$\boldsymbol{x}(t) = M^{-1/2} \left( \sum_{\mu=0}^{n-1} c_\mu \, e^{\pm \mathrm{i}\,(\omega_\mu \, t + \theta_\mu)} \, \boldsymbol{v}_\mu \right). \tag{5}$$

- The phase $\theta_\mu$ cannot be determined by EoM.
  $\Rightarrow$ The behaviors of $\boldsymbol{x}(t)$ looks very different when the phase is different.
- What is the relationship between our oscillation model and actual network dynamics?

# Oscillation Energy of Each Node

- What is the relationship between our oscillation model and actual network dynamics?
  $\Rightarrow$ We regard the ocsillation energy of a node as the strength of the node behavior.
- The oscillation energy $E_i$ of node $i$:

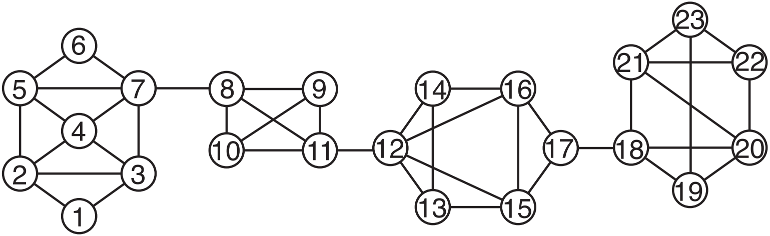
$$E_i := \frac{1}{2} m_i \sum_{\mu=0}^{n-1} \omega_\mu^2 \, |x_i(t)|^2 = \frac{1}{2} \sum_{\mu=0}^{n-1} \omega_\mu^2 \, |y_i(t)|^2$$
$$= \frac{1}{2} m_i \sum_{\mu=0}^{n-1} \omega_\mu^2 \, |a_\mu(t)|^2 \, (v_\mu(i))^2,$$

- Here,

$$\boldsymbol{v}_\mu = {}^t(v_\mu(1), \ldots, v_\mu(n)).$$

# Network Model

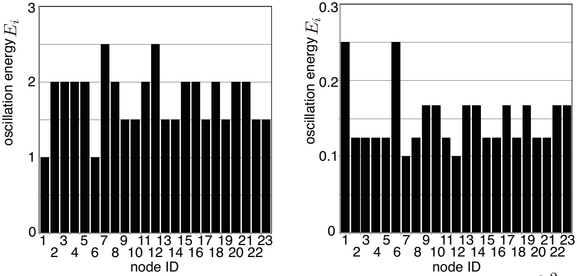
- An example of network for evaluations:




Network with 23 nodes

- The link weights of all links are set to 1.

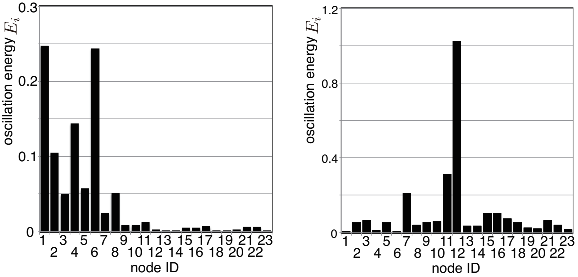
# Oscillation Energy of Node & Node Centrality (1)




(a) $w_{ij} = 1,\ m_i = 1$ (b) $w_{ij} = 1,\ m_i = {d_i}^2$

- When all the oscillation modes contribute at the same strength, $\Rightarrow$ the source of activity is chosen at random.
- In the simplest case (left), the oscillation energy corresponds the degree centrality. The right is an example for directed network. $\Rightarrow$ the oscillation energy gives an extension of the degree centrality.

# Oscillation Energy of Node & Node Centrality (2)




(a) source node $= 1\ (m_i = 1)$ (b) source node $= 12\ (m_i = 1)$

- When a certain node (node 1 or 12) is the source of activity, $\Rightarrow$ the oscillation energy depends on the position of source node.
- By superposing the oscillation energy for all different source nodes, we have the degree centrality.
- The oscillation energy is a generalized notion of the degree centrality (for link direction & the position of source node).

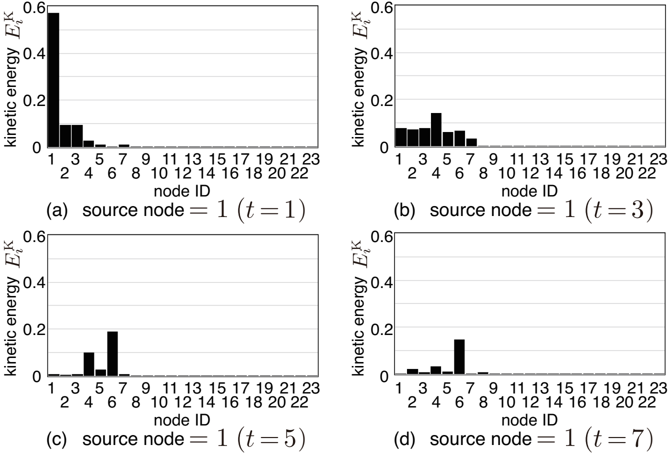
# Kinetic Energy of Each Node

- In order to consider time-dependent measure for node centrality,
  $\Rightarrow$ we introduce the kinetic energy of each node.

$$E_i^{(\mathrm{K})} := \frac{1}{2}\, m_i \left| \sum_{\mu=0}^{n-1} \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} \right|^2 = \frac{1}{2} \left| \sum_{\mu=0}^{n-1} \frac{\mathrm{d}y_i(t)}{\mathrm{d}t} \right|^2$$
$$= \frac{1}{2} \left| \sum_{\mu=0}^{n-1} \frac{\mathrm{d}a_\mu(t)}{\mathrm{d}t}\, v_\mu(i) \right|^2 .$$

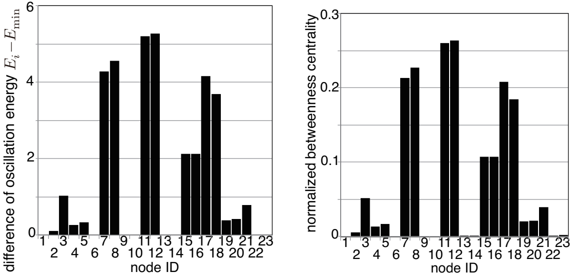
- Time-dependent behavior can be described.
  $\Rightarrow$ It reflects propagation of activity on networks.
- The integrated value of the kinetic energy for a long time is proportional to the oscillation energy.
  $\Rightarrow$ Kinetic energy also gives a generalization of node centrality.

# Kinetic Energy of Each Node




- Temporal evolution of the kinetic energy for the damped oscillation (source node: $1$).

# Oscillation Energy of Node & Betweenness Centrality




(a) difference from minimum energy (b) normalized betweenness centrarity

- The link weight is set as the passing number of shortest paths.
- Oscillation energy for a weighted network (left) & the betweenness centrality (right).
- Oscillation energy is shifted as the minimum energy is zero.
- Our model might be the underlying mechanism for node centrality.

# Damped Oscillation on NWs (1)

- EoM of the damped oscillation on NWs:

$$M\,\frac{\mathrm{d}^2\boldsymbol{x}(t)}{\mathrm{d}t^2}+\gamma\,M\,\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}=-L\,\boldsymbol{x}(t). \tag{6}$$

- EoM of the damped oscillation described using $S$:

$$\frac{\mathrm{d}^2\boldsymbol{y}(t)}{\mathrm{d}t^2}+\gamma\,\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t}=-S\,\boldsymbol{y}(t). \tag{7}$$

- By expanding by eigenbasis of $S$, EoM of each oscillation mode is obtained as

$$\frac{\mathrm{d}^2a_\mu(t)}{\mathrm{d}t^2}+\gamma\,\frac{\mathrm{d}a_\mu(t)}{\mathrm{d}t}+\omega_\mu^2\,a_\mu(t)=0. \tag{8}$$

- By assuming the solution $a_\mu(t)\propto e^{\alpha t}$, the characteristic equation is

$$\alpha^2+\gamma\alpha+\omega_\mu^2=0. \tag{9}$$

  The solution of the damped oscillation is determined by the solution of the characteristic equation.

# Damped Oscillation on NWs (2)

- The solution of the characteristic equation:

$$\alpha = -(\gamma/2) \pm \sqrt{(\gamma/2)^2 - \omega_\mu^2}$$

- If $(\gamma/2)^2 < \omega_\mu^2$, the solution describes damped oscillations:

$$a_\mu(t) = c_\mu \, e^{-(\gamma/2)t} \, e^{\mathrm{i}\sqrt{\omega_\mu^2 - (\gamma/2)^2}\, t + \mathrm{i}\,\theta_\mu}, \tag{10}$$

  where $c_\mu$ and $\theta_\mu$ are constants.
- If $(\gamma/2)^2 = \omega_\mu^2$, the solution describes critical damping:

$$a_\mu(t) = (a_\mu(0) + c_\mu \, t) \, e^{-(\gamma/2)\, t}, \tag{11}$$

  where $c_\mu$ is a constant.
- If $(\gamma/2)^2 > \omega_\mu^2$, the solution describes overdamping:

$$a_\mu(t) = c_\mu^{(1)} \, e^{\alpha_+ t} + c_\mu^{(2)} \, e^{\alpha_- t}, \tag{12}$$

  where $\alpha_+$, $\alpha_-$ are solutions of (9), and $c_\mu^{(1)}$, $c_\mu^{(2)}$ are constants.

# Forced Oscillation on NWs

- Let us apply the external force of frequency $\omega$ to a certain node $j$,

$$M\,\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}+\gamma\,M\,\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t}+L\,\boldsymbol{x}(t)=(F\ \cos\omega t)\,\mathbf{1}_{\{j\}}, \tag{13}$$

whrere $\mathbf{1}_{\{j\}}={}^t(0,\dots,0,\overset{\overset{j}{\vee}}{1},0,\dots,0)$ and $F=$ const.

- By using $\boldsymbol{y}=M^{1/2}\,\boldsymbol{x}$, EoM can be written in the form of

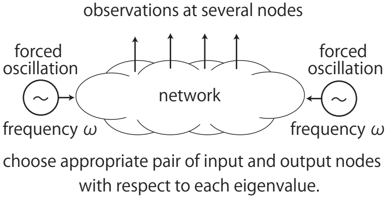
$$\frac{\mathrm{d}^2\boldsymbol{y}(t)}{\mathrm{d}t^2}+\gamma\,\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t}+S\,\boldsymbol{y}(t)=\frac{F\ \cos\omega t}{\sqrt{m_j}}\,\mathbf{1}_{\{j\}}. \tag{14}$$

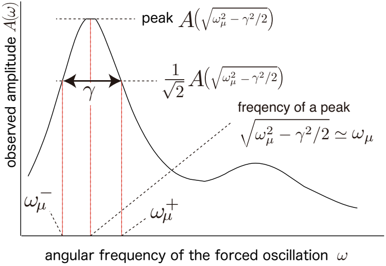
- Let the solution of each mode be $a_\mu(\omega,t)=A_\mu(\omega)\cos(\omega t+\theta_\mu)$, we have

$$A_\mu(\omega)=\frac{F\,b_\mu}{\sqrt{m_j}}\,\frac{1}{\sqrt{(\omega_\mu^2-\omega^2)^2+(\gamma\,\omega)^2}},\ \ \tan\theta_\mu=\frac{-\gamma\,\omega}{\omega_\mu^2-\omega^2}. \tag{15}$$

# Concept of the Network Resonance Method

- Let us give forced oscillation at a certain node and observe its influence at several observation nodes.

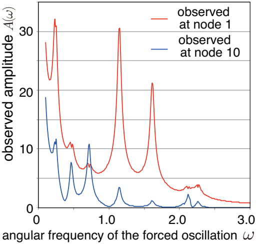


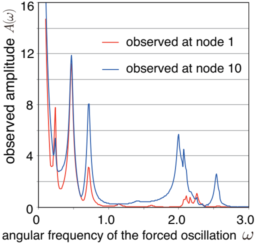

- If $\omega$ of the forced oscillation is near $\omega_\mu$ of eigenfrequency, the amplitude of the oscillation becomes larger (resonanse).
- By observing the behavior of the amplitude with respect to the frequency $\omega$, we can estimate eigenfrequency $\omega_\mu$ (that is eigenvalue $\lambda_\mu = \omega_\mu^2$ of $S$) and the damping factor $\gamma$.

# Example of Amplitude Evaluation for Network Resonance Method

- Behaviors of amplitude for the forced oscillation using the 23-node NW model:



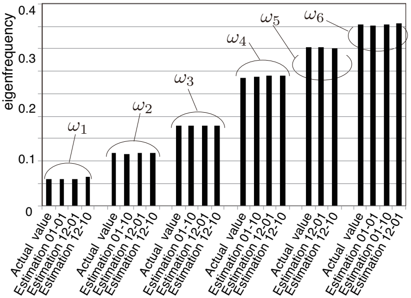
(a) observed amplitude (input node: 1)



(b) observed amplitude (input node: 12)

- If we adopt inappropriate pair of input/output nodes, the peak of the amplitude cannot be observed.
- This problem can be avoided easily by choosing other input/output node pair.

# Example of Eigenfrequency Estimation by Network Resonance Method




- By choosing appropriate pair of input/output nodes, we can estimate eigenfrequencies.
- This means we can estimate the eigenvalues of the original asymmetric Laplacian matrix, that describe the structure of directed networks.

# Conclusions

- Oscillation model for describing propagation of activities on NWs.
  - Oscillation model can describe some kinds of asymmetric link interactions by symmetric matrix.
  - Oscillation energy and kinetic energy give extended notion of the well-known node centralities.

    ⇒ So, the proposed oscillation model might be the underlying mechanism for describing different notions of node centrality by using the common framework.
- Conversely, can we know the structure of the networks by observing the strength of nodes' activity (centrality)?
  - Network resonance method — Eigenvalues can be estimated.

    ⇒ If we also estimate eigenvectors, we can reconstruct the Laplacian matrix.

    ⇒ The estimation of the structure of hidden networks that cannot be observed directly.
  - ex) user networks (strength of friendship, etc.),
    network structure of malicious users that generate cyber attacks, etc.

# References

In details, please see the below:


- Masaki Aida, Chisa Takano and Masayuki Murata,
  "Oscillation model for network dynamics caused by asymmetric node interaction based on the symmetric scaled Laplacian matrix,"
  The 12th International Conference on Foundations of Computer Science (FCS 2016), July 2016.
- Chisa Takano and Masaki Aida,
  "Proposal of new index for describing node centralities based on oscillation dynamics on networks,"
  IEEE GLOBECOM 2016, December 2016.

Thank you for your attention!