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Solvability Analysis and Stabilization of the Cart-Pendulum Modeled by Discrete Mechanics with Friction

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Abstract—This paper deals with the cart-pendulum modeled by discrete mechanics with friction. We analyze the discrete cart-pendulum from the viewpoint of solvability. Then, we consider stabilization of the discrete cart-pendulum via discrete-time optimal regulator theory and propose a method to transform a discrete-time input into a zero-order hold input. Finally, we show some simulations to verify our results.

1. Introduction

Recently, *discrete mechanics* has been researched and attracted attention as a new discretizing technique for mechanical systems [1, 2, 3]. It is known that discrete mechanics has some interesting properties: (i) it can describe energies for conservative/dissipative systems with less errors, (ii) some laws of physics such as Noether's theorem are satisfied. It is expected that discrete mechanics is available for designing controllers with a high affinity for computers. However, there exist few researches on control of mechanical systems via discrete mechanics [4, 5].

We have analyzed the cart-pendulum modeled by discrete mechanics without friction and consider the stabilization problem [8]. However, in order to apply our results to the actual car-pendulum, we have to take account of friction. The aim of this paper is to analyze and stabilize the discrete cart-pendulum considering friction. Section 2 presents a short brief of discrete mechanics. In Section 3, we derive the discrete car-pendulum with friction, and we analyze it from the perspective of solvability of implicit discrete-time nonlinear systems in Section 4. Then, Section 5 gives a control strategy of the discrete cart-pendulum based on discrete-time optimal regulator theory. Finally, a transformation method that changes a discrete input into a zero-order hold input is proposed in Section 6. Some simulations are shown to check the effectiveness.

2. Discrete Mechanics

This section presents basic concepts of discrete mechanics [1, 2, 3]. Let Q be an n-dimensional configuration manifold and $q \in \mathbf{R}^n$ be a generalized coordinate of Q. We also refer to T_qQ as the tangent space of Q at a point $q \in Q$ and $\dot{q} \in T_qQ$ denotes a generalized velocity. Moreover, we consider a time-invariant Lagrangian as $L(q, \dot{q}) : TQ \to \mathbf{R}$. We first explain about the discretization method. The time variable $t \in \mathbf{R}$ is discretized as $t = kh \ (k = 0, 1, 2, \cdots)$ by using a sampling interval $h > 0$. We denote q_k as a point of Q at the time step k, that is, a curve on Q in the continuous setting is represented as a sequence of points $q^d := \{q_k\}_{k=1}^N$ in the discrete setting. The transformation method of discrete mechanics is carried out by the replacement:

$$q \approx (1-\alpha)q_k + \alpha q_{k+1}, \ \ \dot{q} \approx \frac{q_{k+1} - q_k}{h}, \tag{1}$$

where q is expressed as a internally dividing point of q_k and q_{k+1} with a ratio $\alpha \ (0 < \alpha < 1)$. We then define *a discrete Lagrangian*:

$$L_\alpha^d(q_k, q_{k+1}) := hL\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}\right), \tag{2}$$

and *a discrete action sum*:

$$S_\alpha^d(q_0, q_1, \cdots, q_N) = \sum_{k=0}^{N-1} L_\alpha^d(q_k, q_{k+1}). \tag{3}$$

Consider a variation of points on Q as $\delta q_k \in T_{q_k}Q \ (k = 0, 1, \cdots, N)$ with the fixed condition $\delta q_0 = \delta q_N = 0$. In analogy with the continuous setting, by calculating a variation of the discrete action sum (3) and using *the discrete Hamilton's principle*, we obtain *the discrete Euler-Lagrange equations*:

$$\begin{aligned} D_1 L_\alpha^d(q_k, q_{k+1}) + D_2 L_\alpha^d(q_{k-1}, q_k) = 0, \\ k = 1, \cdots, N-1, \end{aligned} \tag{4}$$

where D_1 and D_2 denotes the partial differential operators with respect to the first and second arguments, respectively. It turns out that (4) is represented as difference equations that contains three points q_{k-1}, q_k, q_{k+1}, and we need q_0, q_1 as an initial condition when we simulate (4).

Finally, we explain how to introduce external forces into the discrete Euler-Lagrange equations. By analogy of continuous-time mechanics, we represent a discrete external force as $F^d : Q \times Q \to T^*(Q \times Q)$ and we discretize Lagrange-d'Alembert principle, then we have

$$\delta \sum_{k=0}^{N-1} L_\alpha^d(q_k, q_{k+1}) + \sum_{k=0}^{N-1} F^d(q_k, q_{k+1}) \cdot (\delta q_k, \delta q_{k+1}). \tag{5}$$

Now, we define new discrete external forces F_1, $F_2 : Q \times Q \to T^*Q$ as

$$\begin{aligned} F_1^d(q_k, q_{k+1})\delta q_k &:= F^d(q_k, q_{k+1}) \cdot (\delta q_k, 0), \\ F_2^d(q_k, q_{k+1})\delta q_{k+1} &:= F^d(q_k, q_{k+1}) \cdot (0, \delta q_{k+1}). \end{aligned} \tag{6}$$

The forces F_1 and F_2 above act at the first and second arguments, respectively. Denote a continuous external force as $F^c : TQ \to T^*Q$. We then have the following relations:

$$F_1^d(q_k, q_{k+1}) := (1-\alpha)hF^c\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1}-q_k}{h}\right),$$
$$F_2^d(q_k, q_{k+1}) := \alpha hF^c\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1}-q_k}{h}\right). \tag{7}$$

By (5), (6) can be rewritten as

$$\delta\sum_{k=0}^{N-1} L_\alpha^d(q_k, q_{k+1}) + \sum_{k=0}^{N-1}\{F_1^d(q_k, q_{k+1})\delta q_k + F_2^d(q_k, q_{k+1})\delta q_{k+1}\}. \tag{8}$$

Therefore, we obtain *the discrete Euler-Lagrange equations with external forces* as

$$\begin{aligned} D_1L^d(q_k, q_{k+1}) + D_2L^d(q_{k-1}, q_k) \\ + F_1^d(q_k, q_{k+1}) + F_2^d(q_{k-1}, q_k) = 0, \\ k = 1, \cdots, N-1. \end{aligned} \tag{9}$$

3. Discrete Cart-Pendulum System

We here derive the discrete-time model of the cart-pendulum as shown in Fig. 1. Let $\theta \in \mathbf{S} := (-\pi, \pi]$ be the angle of the pendulum and $z \in \mathbf{R}$ be the position of the cart. We set parameters of the system: the mass of the pendulum m, the mass of the cart M, and the length of the pendulum l. The Lagrangian of this system is given by

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}\dot{z}\cos\theta + \frac{1}{2}(m+M)\dot{z}^2 - mgl\cos\theta, \tag{10}$$

and we then have the discrete Lagrangian with the discrete variables shown in Fig. 1:

$$\begin{aligned} L^d &= \frac{m+M}{2h}(z_{k+1}-z_k)^2 + \frac{ml^2}{2h}(\theta_{k+1}-\theta_k)^2 \\ &+ \frac{ml}{h}\cos\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\}(z_{k+1}-z_k)(\theta_{k+1}-\theta_k) \\ &- mglh\cos\{(1-\alpha)\theta_k + \alpha\theta_{k+1}\}. \end{aligned} \tag{11}$$

In addition, we consider viscous frictions of the pendulum and the cart as depicted in Fig. 1, where η and μ are friction coefficients of the pendulum and the cart, respectively. Consequently, from (9), we derive the discrete Euler-Lagrange equation of the cart-pendulum with friction as (12) and (13).

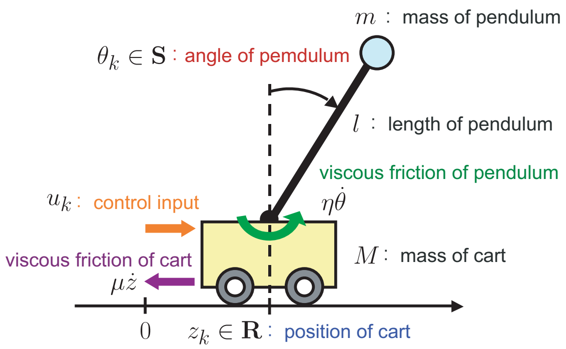

Fig. 1 : Cart-Pendulum System

Note that we add the discrete frictions to the discrete Euler-Lagrange equation by using discrete Lagrange-d'Alembert principle (7) and (8). Substituting $\theta_{k-1} = \theta_k = \theta_{k+1}$, $z_{k-1} = z_k = z_{k+1}$ and $u_k = 0$ into (12) and (13), we have $\sin\theta_k = 0$. Therefore, the equilibria of the discrete cart-pendulum are $(\theta_k, z_k) = (0, z^e)$, (π, z^e), $\forall z^e \in \mathbf{R}$, that is, they correspond with those of the usual cart-pendulum in the continuous setting. Finally, we calculate the linear approximation system that behaves around the equilibrium $\theta_k = 0$. Considering θ_{k-1}, θ_k, $\theta_{k+1} \approx 0$ for (12) and (13), we obtain the linear approximation as (14) and (15).

4. Solvability Analysis

By using appropriate functions f and g, we can rewrite (12) and (13) as

$$f(\theta_{k-1}, \theta_k, \theta_{k+1}, z_{k-1}, z_k, u_k) = 0, \tag{16}$$
$$z_{k+1} = g(\theta_{k-1}, \theta_k, \theta_{k+1}, z_{k-1}, z_k, u_k). \tag{17}$$

We can see that (17) is explicit for z_{k+1}, however, (16) is implicit for θ_{k+1}. In general, systems modeled by discrete mechanics contain some implicit equations, hence we have to treat implicit nonlinear discrete-time systems. In this section, we investigate solvability of the discrete cart-pendulum system. We first explain some concepts of solvability for implicit nonlinear discrete-time systems:

$$f_k(x_k, x_{k+1}, u_k) = 0, \quad k = 0, 1, \cdots, N-1, \tag{18}$$

where $k \in \{0, \cdots, N\}$ is a time step, $x_k \in \mathbf{R}^n$ is a state, $u_k \in \mathbf{R}^m$ is an input and $f_k : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^r$ is a nonlinear function. We can find examples of implicit nonlinear discrete-time systems in economics [6]. The class of implicit nonlinear discrete-time systems contains descriptor systems and is the largest in all of the discrete-time control systems. Luenberger [6] and Fliegner et al. [7] discussed *solvability* for such systems. If a given pair of a state sequence $x := (x_0, x_1, \cdots, x_N)$ and an input sequence $u := (u_0, u_1, \cdots, u_{N-1})$ satisfies all the equations of (18), it is called *admissible*. *A solvability matrix* for a admissible pair (x, u) is defined by

$$F_{(x,u)}(0, N) := \begin{bmatrix} G_0 & H_1 & & & \\ & G_1 & H_2 & & \\ & & \ddots & & \\ & & & G_{N-1} & H_N \end{bmatrix},$$
$$G_i := \left.\frac{\partial f_i}{\partial x_i}\right|_{(x_i, x_{i+1}, u_i)}, \quad i = 0, \cdots, N-1,$$
$$H_i := \left.\frac{\partial f_i}{\partial x_{i+1}}\right|_{(x_{i-1}, x_i, u_{i-1})}, \quad i = 1, \cdots, N. \tag{19}$$

Solvability of the system (18) is defined as follows [7].

Definition 1: The implicit nonlinear discrete-time system (18) is said to be *solvable* if the solvability matrix (19) has a row full-rank for any admissible pair (x, u). ■

In the simplest terms, solvability means the existence of x_{k+1} for given x_k and u_k. In order to check solvability of

$$
\begin{aligned}
&-ml(1-\alpha)(\theta_{k+1}-\theta_k)(z_{k+1}-z_k)\sin\{(1-\alpha)\theta_k+\alpha\theta_{k+1}\} - ml\cos\{(1-\alpha)\theta_k+\alpha\theta_{k+1}\}(z_{k+1}-z_k) - ml^2(\theta_{k+1}-\theta_k)\\
&\quad + mgl(1-\alpha)h^2\sin\{(1-\alpha)\theta_k+\alpha\theta_{k+1}\} + ml\cos\{(1-\alpha)\theta_{k-1}+\alpha\theta_k\}(z_k-z_{k-1}) + ml^2(\theta_k-\theta_{k-1})\\
&\qquad - ml\alpha(\theta_k-\theta_{k-1})(z_k-z_{k-1})\sin\{(1-\alpha)\theta_{k-1}+\alpha\theta_k\} + mgl\alpha h^2\sin\{(1-\alpha)\theta_{k-1}+\alpha\theta_k\}\\
&\qquad\quad + \eta\{(1-\alpha)(\theta_{k+1}-\theta_k)+\alpha(\theta_k-\theta_{k-1})\} = 0
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&-(m+M)(z_{k+1}-z_k) - ml(\theta_{k+1}-\theta_k)\cos\{(1-\alpha)\theta_k+\alpha\theta_{k+1}\} + (m+M)(z_k-z_{k-1})\\
&\quad + ml(\theta_k-\theta_{k-1})\cos\{(1-\alpha)\theta_k+\alpha\theta_{k-1}\} + \mu\{(1-\alpha)(z_{k+1}-z_k)+\alpha(z_k-z_{k-1})\} + hu_k = 0
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&-ml(z_{k+1}-z_k) + mgl(1-\alpha)h^2\{(1-\alpha)\theta_k+\alpha\theta_{k+1}\} + ml(z_k-z_{k-1}) - ml^2(\theta_{k+1}-\theta_k) + ml^2(\theta_k-\theta_{k-1})\\
&\quad + mgl\alpha h^2\{(1-\alpha)\theta_{k-1}+\alpha\theta_k\} + \eta\{(1-\alpha)(\theta_{k+1}-\theta_k)+\alpha(\theta_k-\theta_{k-1})\} = 0
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&-(m+M)(z_{k+1}-z_k) - ml(\theta_{k+1}-\theta_k) + ml(\theta_k-\theta_{k-1}) + (m+M)(z_k-z_{k-1})\\
&\quad + \mu\{(1-\alpha)(z_{k+1}-z_k)+\alpha(z_k-z_{k-1})\} + hu_k = 0
\end{aligned}
\tag{15}
$$

---

a given system, *the shuffle algorithm* based on the implicit theorem is introduced [7]. We can investigate solvability by the rank of a finally obtained system in the algorithm. Because of space limitations, we omit its details (see [7]). We now check solvability of the discrete cart-pendulum (12) and (13). The next can be proven by the shuffle algorithm.

Proposition 1: Assume that the sampling time is sufficient small, i.e., $h \ll 1$. Then, the discrete cart-pendulum with friction (12) and (13) is solvable at any point (θ, z). ∎

We next consider solvability when the system behaves around the equilibrium $\theta_k = 0$ and the sampling time h is not subject to restrictions. The following can be derived.

Proposition 2: The discrete cart-pendulum with friction (12) and (13) is solvable around neighborhood of the equilibrium point $(0, z)$ if

$$
\begin{aligned}
&4ml\{Ml-(m+M)g\alpha(1-\alpha)h\} + \eta\mu h\\
&\quad + 2\eta h(m+M) - 2ml\mu h\{l - g\alpha(1-\alpha)h\} \neq 0
\end{aligned}
\tag{20}
$$

holds for its parameters. ∎

From Proposition 2, if the sampling time h satisfies (20) the discrete cart-pendulum with friction (12) and (13) is always solvable around the equilibrium $\theta_k = 0$. Since Proposition 2 is a sufficient condition, the system has a possibility of solvability though (20) fails.

5. Stabilization of Discrete Cart-Pendulum

This section gives a stabilizing controller for the discrete cart-pendulum. We first set a state variable as $x_k = [\,x_k^1\ x_k^2\ x_k^3\ x_k^4\,]^\mathsf{T} = [\,\theta_{k-1}\ \theta_k\ z_{k-1}\ z_k\,]^\mathsf{T}$. From (14) and (15), we then obtain the discrete-time linear control system with appropriate matrices $A \in \mathbf{R}^{4\times4}$, $B \in \mathbf{R}^{4\times1}$:

$$x_{k+1} = Ax_k + Bu_k. \tag{21}$$

By solving discrete-time optimal regulator problem for (21), we can design a controller in the form

$$u_k = Kx_k, \tag{22}$$

where $K \in \mathbf{R}^{1\times4}$ is a gain matrix. We use the algorithm proposed in [8] to stabilize the discrete cart-pendulum .

We now show simulations. The parameters are set as $m = 0.035$ [kg], $M = 1.038$ [kg], $l = 0.12$ [m], $\eta = 1.2\times10^{-5}$ [Nms/rad], $\mu = 15.11$ [Ns/m] $\alpha = 0.5$. The weight matrices for the discrete-time optimal regulator problem are set as $Q = \mathrm{diag}(1, 0, 1.0, 5.0, 5.0)$, $R = 0.005$. The initial states are given as $\theta_0 = \pi/6$ [rad], $\theta_1 = \pi/6$ [rad], $z_0 = 0.1$ [m], $z_1 = 0.1$ [m]. Fig. 2 and Fig. 3 show the time responses of θ and z with the sampling times $h = 0.02$ [s] and 0.1 [s], respectively. From these, it is confirmed that the discrete cart-pendulum with friction is stabilized at not only $h = 0.02$ but also a large sampling time $h = 0.1$. Therefore, we can say that it is appropriate to design controllers for systems based on their linear approximations in discrete mechanics.

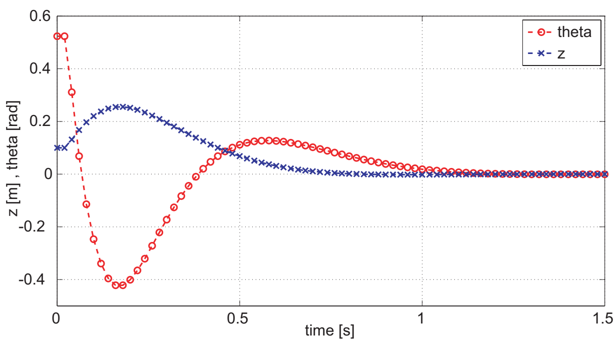

Fig. 2 : Time Responses of θ_k and z_k ($h = 0.02$)

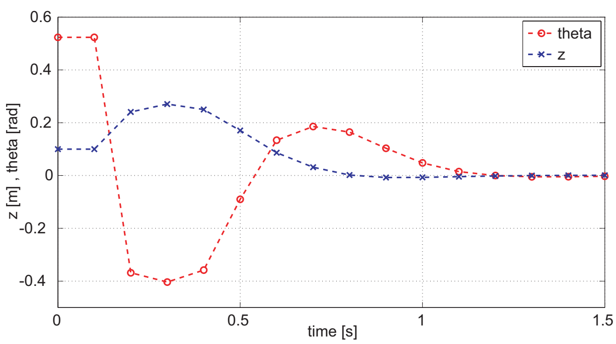

Fig. 3 : Time Responses of θ_k and z_k ($h = 0.1$)

6. Construction of Zero-Order Hold Input

In Section 5, we have shown a discrete stabilizing controller for the discrete cart-pendulum, and its effectiveness by simulations. However, in order to apply the discrete controller to the actual cart-pendulum, we have to consider inputs between sampling points. We here propose a method that transforms a discrete input into a zero-order hold input:

$$u^c(t) = Lx_k, \quad kh \leq t < (k+1)h, \tag{23}$$

where $L \in \mathbf{R}^{1\times 4}$ is a gain matrix. That is, (23) implies a state feedback law using the value of x_k during $kh \leq t < (k+1)h$. The following states that the gain L in (23) can be determined from the gain K of the discrete input in (22).

Proposition 3: By discrete Lagrange-d'Alembert principle (7) and (8), the gain matrix of the zero-order hold input is obtained as

$$L = \frac{K}{h} \tag{24}$$

from the discrete input (22). ∎

Now, we show a simulation of stabilization for the normal cart-pendulum. We use the same parameters of Section 5, and set the initial states and the sampling time as $\theta(0) = \pi/6\,[\mathrm{rad}]$, $\dot{\theta}(0) = 0\,[\mathrm{rad/s}]$, $z(0) = 0.1\,[\mathrm{m}]$, $\dot{z}(0) = 0\,[\mathrm{m/s}]$, $h = 0.04\,[s]$. Fig. 4 shows the zero-order hold input derived by (23) and (24). In Fig. 5, the time responses of θ and z are depicted, and it can be confirmed that the continuous cart-pendulum is stabilized by the proposed zero-order hold input. Therefore, we can say that our proposed method is available for the normal cart-pendulum.

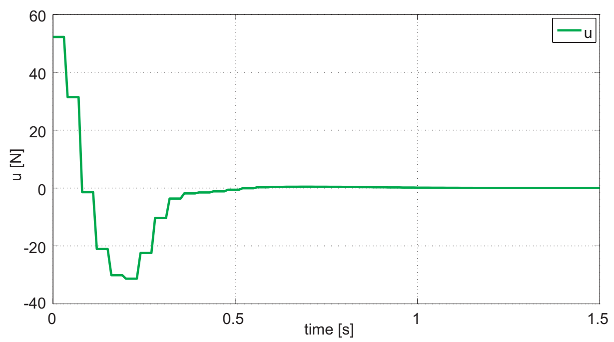

Fig. 4 : Zero-Order Hold Input

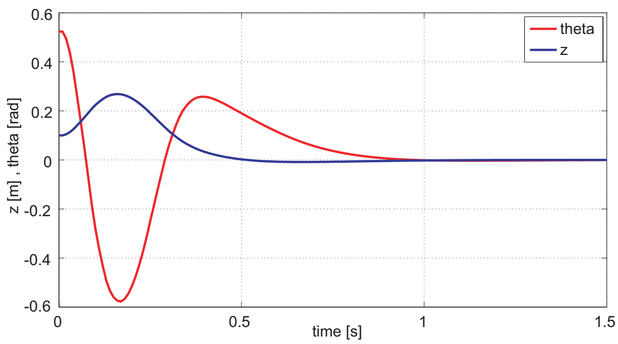

Fig. 5 : Time Responses of θ and z

7. Conclusion

In this paper, we have considered control problems for the cart-pendulum systems with friction, which is modeled by discrete mechanics. We have given solvability analysis for the discrete cart-pendulum. Then, we have proposed a controller to stabilize the system by optimal regulator theory and a transformation method to obtain a zero-order hold input. Some simulation results have indicated effectiveness of the proposed controller. Our future work are as follows: (i) nonlinear control laws and swing-up control for the discrete cart-pendulum, (ii) applications of our results to the actual cart-pendulum.

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