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Nonlinear Control Design via Formal Linearization of Polynomial Type Using Taylor Expansion

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Abstract—This paper is concerned with a nonlinear feedback control problem. The purpose of this paper is to design a nonlinear feedback control for single input nonlinear systems, which is effective in wider region in state space, using a formal linearization method. A formal linearization function which is composed of polynomials is introduced, and then a given nonlinear system is transformed into an augmented bilinear system with respect to a formal linearization function. In order to linearize it, we exploit the ordinary LQ control technique which is derived by a linearization of Taylor expansion truncating at the first order. As a result, we synthesize a nonlinear feedback control for the given nonlinear system by applying a linear system theory. Numerical experiments are illustrated to show effectiveness of this method.

1. Introduction

When we deal with the nonlinear feedback control problem, one might face with difficulties because of the complexity of nonlinearity. Many nonlinear tools have been developed, however most of them can be the tools for some particular nonlinear systems(e.g. [1, 2, 3, 4]). So we need to employ the fit tool that is one of the best appropriate ones for the given nonlinear problem. We cannot expect one particular procedure to apply all nonlinear systems. On the other hand, there are a few tools which cover a wide range of nonlinear problems (e.g. [5, 6, 7, 8]). Formal linearization method [9, 10, 11, 12, 13, 14] is one of them.

We have been studying the formal linearization method and tried to apply the method to nonlinear control problems. In the previous work [14], we have presented a nonlinear control design for single input nonlinear systems by using two-stage formal linearization and two-type LQ controls. This method is easily applicable to nonlinear systems, but has complexities of linearization and some problems like how to select parameters to transform into a formal linear system. In this paper, we consider a simple nonlinear control design for single input nonlinear systems using a formal linearization method. A formal linearization function which is composed of polynomials up to higher order is introduced. Deriving the derivative of the formal linearization function along with the solution of a given nonlinear system and expanding a nonlinear function by Taylor expansion up to higher order, a given nonlinear system is transformed into an augmented bilinear system with respect to a formal linearization function and a control. Exploiting the ordinary LQ control which is obtained by a linearization of Taylor expansion truncating at the first order and substituting it to the control of augmented elements in the bilinear system, a formal linear system is obtained with respect to the formal linearization function. If the formal linear system is controllable, or at least stabilizable, we can find the state feedback control which stabilize the given nonlinear system.

This method is simple to design a nonlinear control and directly applicable to a nonlinear feedback problem. We will illustrate numerical examples to show simpleness and effectiveness of this method.

2. Statement of Problem

For the sake of simplicity, we consider a nonlinear control problem using a formal linearization method for scalar systems. For vector systems, it is straightforward. We consider a class of nonlinear systems of the form

$$\Sigma_1 : \dot{x}(t) = f\big(x(t)\big) + bu, \ x \in D, \tag{1}$$

where $t > 0$ denotes time, overdot represents derivative with respect to t, x is a state variable, D is domain, $f \in C$ is a nonlinear function with $f(0) = 0$, b is a constant and u is an input.

3. Nonlinear Control by Formal Linearization

To design nonlinear feedback control, we exploit a formal linearization method [9, 10, 11, 12, 13, 14] and apply Taylor expansion truncating up to the N-th order to approximate a given nonlinear system to a formal linear system.

A formal linearization function is defined as

$$\begin{aligned}\boldsymbol{\phi}(x) &= [x,\ x^2,\ x^3, \cdots,\ x^N]^T \qquad (2)\\ &= [\phi_1(x),\ \phi_2(x),\ \phi_3(x), \cdots,\ \phi_N(x)]^T\end{aligned}$$

where T denotes transpose. The derivative of the element of $\boldsymbol{\phi}$ is

$$\begin{aligned}\dot{\phi}_i(x) &= ix^{i-1}\dot{x} \\ &= ix^{i-1}\Big(f(x)+bu\Big) \quad (i=1,2,\cdots,N).\end{aligned} \tag{3}$$

Applying Taylor expansion to the nonlinear function $f(x)$ in Eq.(3) about $x=0$

$$\dot{\phi}_i(x) = ix^{i-1}\Big(f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \cdots + bu\Big)$$

where

$$f^{(i)}(0) = \frac{\partial^i}{\partial x^i}f(x)|_{x=0} \ .$$

Truncating up to the N-th order yields

$$\begin{aligned}\dot{\phi}_i(x) &\approx if'(0)x^i + \frac{i}{2!}f''(0)x^{i+1} + \frac{i}{3!}f^{(3)}(0)x^{i+2} + \\ &\quad \cdots + \frac{i}{(N-i+1)!}f^{(N-i+1)}(0)x^N + ibx^{i-1}u \\ &= if'(0)\phi_i + \frac{i}{2!}f''(0)\phi_{i+1} + \frac{i}{3!}f^{(3)}(0)\phi_{i+2} + \\ &\quad \cdots + \frac{i}{(N-i+1)!}f^{(N-i+1)}(0)\phi_N + ib\phi_{i-1}u \ .\end{aligned} \tag{4}$$

So a bilinear system with respect to the formal linearization function is derived by

$$\dot{\boldsymbol{\phi}}(x) = \bar{A}\boldsymbol{\phi}(x) + b\begin{pmatrix} 1 \\ 2\phi_1 \\ \vdots \\ N\phi_{N-1}\end{pmatrix}u \tag{5}$$

where

$$[\bar{A}_{ij}] = \begin{cases} \Big[\frac{i}{(j-i+1)!}f^{(j-i+1)}(0)\Big] & (i \le j) \\ [0] & (i > j)\end{cases} ,$$

$$(i, j = 1, 2, \cdots, N) \ .$$

In order to transform the bilinear system (Eq.(5)) into a linear system, we use the ordinary LQ control [15] for the linearized system which is obtained by Taylor expansion truncating at the first order about the origin

$$\Sigma_0 : \dot{x} = A_0 x + bu \tag{6}$$

where

$$A_0 = f'(0) = \frac{\partial}{\partial x}f(x)|_{x=0} \ .$$

Assume the pair (A_0, b) is controllable, or at least stabilizable. Let a cost function be

$$J_0 = \int_0^\infty (Q_0 x^2 + R_0 u^2)dt \tag{7}$$

where $Q_0 \ge 0$ and $R_0 > 0$. Applying the LQ control theory to this linearized system in Eqs.(6) and (7) yields

$$u_0(x) = -R_0^{-1}bP_0 x \tag{8}$$

where $P_0 > 0$ satisfies the Riccati equation

$$2P_0A_0 + Q_0 - P_0^2 b^2 R_0^{-1} = 0 \ . \tag{9}$$

Substitute this linear control u_0 in Eq.(8) into the control u of augmented elements in the bilinear system (Eq.(4))

$$\begin{aligned}\dot{\phi}_i(x) &= i\Big(f'(0) - b^2\frac{P_0}{R_0}\Big)\phi_i + \frac{i}{2!}f''(0)\phi_{i+1} + \frac{i}{3!}f^{(3)}(0)\phi_{i+2} + \\ &\quad \cdots + \frac{i}{(N-i+1)!}f^{(N-i+1)}(0)\phi_N, \ (i = 2, 3, \cdots N).\end{aligned} \tag{10}$$

And the bilinear system (Eq.(5)) is approximated to a linear system

$$\dot{\boldsymbol{\phi}}(x) = A\boldsymbol{\phi}(x) + cu \tag{11}$$

where

$$[A_{ij}] = \begin{cases} \Big[\frac{i}{(j-i+1)!}f^{(j-i+1)}(0)\Big] & (i < j) \\ \Big[if'(0) - ib^2\frac{P_0}{R_0}\Big] & (i = j) \\ [0] & (i > j)\end{cases} ,$$

$$c = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus the formal linear system is obtained as

$$\begin{aligned}\Sigma_2 : \dot{z}(t) &= Az(t) + cu(t) \ , \\ z(0) &= \boldsymbol{\phi}(x(0)) \ .\end{aligned} \tag{12}$$

Its inversion is simply obtained from Eq.(2) by

$$\hat{x}(t) = [1\ 0\ 0 \cdots\ 0]\boldsymbol{\phi}(x(t)) = [1\ 0\ 0 \cdots\ 0]z(x(t)) \ . \tag{13}$$

If the formal linear system (Eq.(12)) is controllable, or at least stabilizable, we can find the state feedback control which stabilize the state variable $\boldsymbol{\phi}$, namely the original state variable x. Let a cost function be

$$J = \int_0^\infty (z^T Q z + Ru^2)dt \tag{14}$$

where $Q \ge 0$ and $R > 0$. By an application of the LQ control theory to the system in Eqs.(12) and (14) yields

$$\hat{u}(t) = -R^{-1}c^T P z(t) \tag{15}$$

where $P > 0$ satisfies the Riccati equation

$$PA + A^T P + Q - PcR^{-1}c^T P = 0. \tag{16}$$

The control $\hat{u}$ in Eq.(15) is a nonlinear feedback control which can stabilize the given nonlinear system (Eq.(1)). Thus the closed-loop system becomes

$$\dot{x}(t) = f\Big(x(t)\Big) + b\hat{u}(t) \ . \tag{17}$$

4. Numerical Experiments

Numerical experiments are illustrated by the following simple nonlinear control system. Consider the system

$$\dot{x}(t) = x^2(t) + u \,, \quad (18)$$

$$x \in [0, \infty) \,.$$

This nonlinear system (Eq.(18)) is transformed into a bilinear system (Eq.(5)) with respect to the formal linearization function $\boldsymbol{\phi}$ of Eq.(2). When the order of $\boldsymbol{\phi}$ is $N = 3$, the bilinear system is

$$\dot{\boldsymbol{\phi}}(x) = \frac{\partial}{\partial t}\begin{pmatrix} x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\boldsymbol{\phi}(x) + \begin{pmatrix} 1 \\ 2\phi_1 \\ 3\phi_2 \end{pmatrix} u \,. \quad (19)$$

In order to investigate the accuracy of this linearization, we show the trajectories of the state variable $\hat{x}$ for a system $\dot{\boldsymbol{\phi}}(x) = A\boldsymbol{\phi}(x)$ when $u = 0$, namely an autonomous system

$$\dot{x}(t) = x^2(t) \quad (20)$$

and its approximated values $\hat{x}$ obtained by inversion

$$\hat{x}(t) = [1\ 0\ 0 \cdots\ 0]\boldsymbol{\phi}(x(t)). \quad (21)$$

Fig. (1) shows a true value $x(t)$ and $\hat{x}(t)$ when the order of $\boldsymbol{\phi}(x)$ is varied as $N = 1$ to 6 and an initial value is $x(0) = 0.1$.

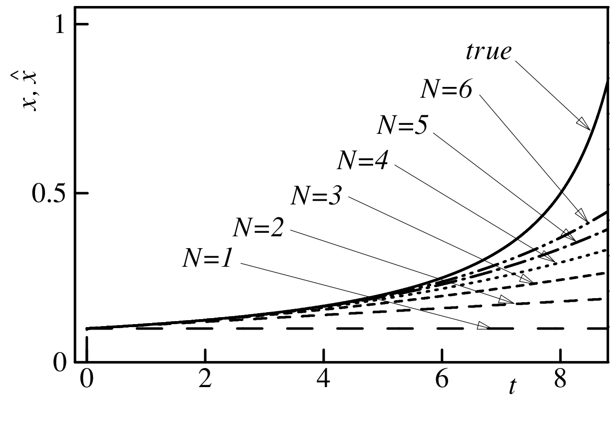

Figure 1: x and $\hat{x}$ of an autonomous system by the formal linearization

In order to transform the bilinear system (Eq.(19)) into the formal linear system (Eq.(12)), we set u_0 in Eq.(8). Linearization of the given system (Eq.(18)) at the origin results in the linear system

$$\dot{x}(t) = u \quad (22)$$

and the LQ control for the system is

$$u_0 = -x \quad (23)$$

when the parameters are

$$Q_0 = 1, R_0 = 1$$

in the Riccati equation (Eq.(9)). Substituting the linear control u_0 in Eq.(23) into u of the second and third order elements of the bilinear system (Eq.(19)) yields the linear system

$$\dot{\boldsymbol{\phi}}(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{pmatrix}\boldsymbol{\phi}(x) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u \,. \quad (24)$$

To the linear system (Eq.(24)), we can apply the LQ control theory. Let the parameters in Eq.(14) be

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ R = 1 \,.$$

Solving the Riccati equation (Eq.(16)) yields the positive definite matrix

$$P = \begin{pmatrix} 1 & 0.333 & 0.167 \\ 0.333 & 0.389 & 0.178 \\ 0.167 & 0.178 & 0.281 \end{pmatrix}.$$

Thus the nonlinear feedback control for the given nonlinear system becomes

$$\hat{u}(t) = [-1,\ -0.333,\ 0.167]\boldsymbol{\phi}(x) = -x - 0.333x^2 - 0.167x^3. \quad (25)$$

Fig. (2) shows results of time responses of the closed-loop system (Eq.(17)) at $x(0) = 1.4$ when the order of the formal linear system is varied as $N = 1$ to 6.

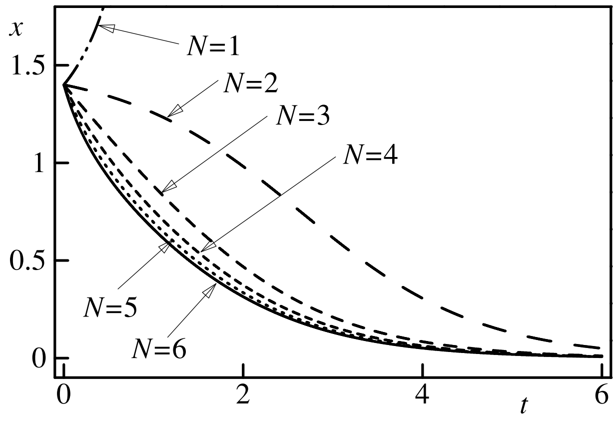

Figure 2: Results for stabilizing nonlinear system

When the order of the formal linearization function is $N = 1$, the formal linear system is the same as the ordinary LQ control system by Taylor expansion truncating at the first order. It means that the proposed method can stabilize the system even in the region in which the conventional method can not stabilize. Table I shows values of the cost function

$$J = \int_0^\infty (x^2 + \hat{u}^2)dt \quad (26)$$

when the order of the formal linear system is varied as $N =$ 1 to 6 and the performance is improved as N increases.

Table I
Values of cost function

	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
J	$\times$	11.604	6.349	5.522	5.186	5.011

($\times$: Very large value)

Next we make a comparison between our method (New method) and the previous method [14] (Old method). Table II shows stable regions from the origin and cost functions J in Eq.(26) when the order of the formal linearization function is $N = 6$ and parameters in the previous one are the same in the paper [14].

Table II
Comparison with the previous method

	Stable region	J
New method	7.44	5.01
Old method	5.00	12.03

5. Conclusions

This paper has considered a design of nonlinear feedback control for nonlinear systems using a formal linearization method. This approach is simple to design and easily applicable to a nonlinear feedback problem. Numerical experiments show that the proposed approach is effective in stabilizing a nonlinear feedback control problem and can improve the performance of the nonlinear control as the order of the formal linear system is increased. And it also indicates that the proposed method is better than the previous work.

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