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Bifurcation Analysis of Chaos Synchronization in Coupled BVP Oscillators

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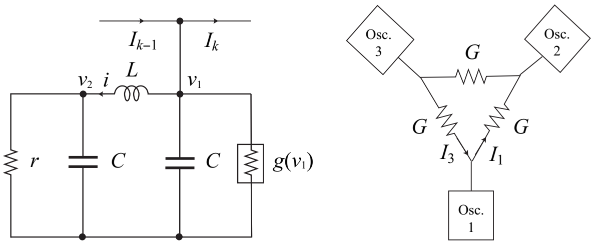
Abstract—In this paper, we investigate three chaotic BVP oscillators which are resistively coupled as a ring. Since all parameters of oscillators are identical, attractors appeared in the system have a strong symmetrical properties observed in the individual oscillator. We found completely in-phase synchronization for periodic solutions and chaos. Also 3-phase synchronized tori and self-switching synchronized chaos. We show some bifurcation diagrams to explain the parameter regions exhibiting these attractors.

1. Introduction

Coupled oscillators are frequently used to simulate biological and mathematical objects since they are easily configured by linear and nonlinear elements[1], and corresponding differential equations can be simulated with CAD system and bifurcation analysis tools. Especially a possibility for implementation of their real circuit is very attractive for researchers. With recent technologies, they are able to be implemented into a small circuit board, or tiny silicon chip. The whole dynamical behavior of the given coupled oscillators are determined not only by oscillator dynamics but also its coupling structure. In the mathematical descriptions of coupling oscillators, this structure reflects into symmetry in the equations. The Bonhöffer-van der Pol (BVP) oscillator is known as a model for a neuron unit. It is written by a two-dimensional nonlinear differential equation, and can be realized by a simple circuit implementation. Its extensional circuits also exhibit a rich variety of nonlinear phenomena, for example, as a model of chemical reactions in an oil-water interface, Yoshinaga studied a three-dimensional model given by adding a coil and resistor[2], and it includes chaotic behavior. Nishiuchi proposed another extended BVP oscillator including an extra capacitor on the way of studying coupled BVP oscillators[3]. Stable tori are caused by Neimark-Sacker bifurcation, and chaos attractors are obtained via a torus breakdown scenario. While synchronization phenomena is a remarkable keyword of nonlinear systems, especially chaos synchronization has been studied intensively for a decade because realization of simple secure communication system is expected with the chaos synchronization. In this paper, we choose the extended BVP oscillator, which has been studied by Nishiuchi, as an oscillation unit, and study their coupled system. We assume that all oscillators are identical, and coupling conductors are symmetric. Thus the dynamical properties reflect these symmetrical characteristics, that is, complete in-phase synchronization phenomena are expected. Since synchronization of two coupled chaotic oscillators have been investigated in the preceded papers[4][5] thus we investigate three coupled oscillators as a next step toward understanding synchronization properties of large-scale complex dynamical systems. In this paper, we show bifurcation diagrams of equilibria, limit cycles, and synchronized periodic orbits. As an result, the parameter regions exhibiting stable chaos synchronization are specified in the parameter plane.

2. Circuit Model



(a) (b)

Figure 1: Circuit model. (a) The k-th oscillator, (b) coupled oscillators

Figure 1 shows the extended BVP circuit as a unit oscillator and coupled oscillators. Let us assume that the coupling conductors have the same value G, and all oscillators are identical. The circuit equation for Fig.1(b) is described by

$$
\begin{cases}
C\dfrac{dv_{1,j}}{dt} & = & -i_j - g(v_{1,j}) \\
& & +G(v_{1,j+1} + v_{1,j-1} - 2v_{1,j}) \\
C\dfrac{dv_{2,j}}{dt} & = & i_j - \dfrac{v_{2,j}}{r} \\
L\dfrac{di_j}{dt} & = & v_{1,j} - v_{2,j}
\end{cases}
\tag{1}
$$

where $j = 1, 2, 3$ (modulo 3), and the nonlinear resistor $g(v)$

is approximated by a cubic characteristics:

$$g(v) = -a\left(v - \frac{v^3}{3}\right) \tag{2}$$

With the variable transformations

$$\cdot = \frac{d}{d\tau}, \quad \tau = \frac{t}{\sqrt{LC}},$$
$$x_j = \sqrt{C}v_{1,j}, \quad y_j = \sqrt{C}v_{2,j}, \quad z_j = \sqrt{L}i_j,$$
$$k = \frac{r}{\sqrt{L}}, \quad \delta = G\sqrt{\frac{L}{C}}, \quad \gamma = \frac{1}{C}\sqrt{\frac{L}{C}},$$

then we have the normalized nine-dimensional differential equation for Eq. (1):

$$\begin{cases} \dot{x}_j &= -z_j + \gamma\left(x_j - \frac{x_j^3}{3}\right) + \delta(x_{j+1} + x_{j-1} - 2x_j) \\ \dot{y}_j &= z_j - \frac{y_j}{k} \\ \dot{z}_j &= x_j - y_j \end{cases} \tag{3}$$

where, $j = 1, 2, 3$. Before numerical computation, the following factors are found from the form of Eq. (3) analytically:

- The origin $O \in \boldsymbol{R}^9$ is an equilibrium.
- Each oscillator has a symmetrical property; $(x_j, y_j, z_j) \rightarrow (-x_j, -y_j, -z_j)$.
- Let $\boldsymbol{x}_j = (x_j, y_j, z_j)$ be an local coodinate of the state in each oscillator. If we put exactly same initial values to $\boldsymbol{x}_j(t)$ for all oscillators, a completely in-phase synchronization is achieved for all time since the coupling terms are canceled.

We compute bifurcation sets of equilibria and periodic solutions by using the BunKi package[6] developed partially by the authors.

3. Bifurcation of equilibria in k-γ plane

Firstly we fix $\delta = 0.1$. Since all oscillator are identical and the coupling structure retains symmetry, an equilibrium in the coupled system is given by a three-tuple of local coordinate values (x^*, y^*, z^*) for each uncoupled single oscillator, i.e.,

$$\boldsymbol{x}^* = (x^*, y^*, z^*, x^*, y^*, z^*, x^*, y^*, z^*). \tag{4}$$

Figure 2 shows a bifurcation diagram in k-γ plane for the whole coupled system. In Region (A), we only have a stable limit cycle generated by Hopf bifurcation of equilibria (it is out of this diagram), and the unstable origin. As parameter varies along the direction of the arrows on the curve labeled $d + h_1$, we have two symmetric stable equilibria C^+ and C^- satisfying $(x^*, y^*, z^*) \rightarrow (-x^*, -y^*, z^*)$, and two symmetric limit cycles around $C^\pm$. Different type of Hopf bifurcation is happened for $C^\pm$ on h_2. Thus we have two different symmetric stable limit cycles around $C_\pm$ in Region (C).

4. Bifurcation of synchronized attractors in k-γ plane

Now we are focus on completely in-phase periodic solutions. Symbols I, G, and NS show period-doubling, tangent, and Neimark-Sacker bifurcations of a limit cycle, respectively. I^2 means a period-doubling bifurcation of 2-periodic solutions. From Region (D) to (C), one of above symmetric limit cycles (Fig. 3 (a)) meet Neimark-Sacker bifurcation, and a torus is obtained after that. Other type of symmetric limit cycles become chaos attractor via the period-doubling cascade (Fig. 3 (b)). Furthermore, parameters are changed, these chaos attractor adhere and it becomes one chaos attractor (Fig. 3 (c)).

An uncoupled oscillator definitely has the same attractors with same parameter set of k-γ since completely in-phase means no coupling, but stability of them cannot be determined. We emphasize that these attractors introduced in this section are completely in-phase, and stable. This means that chaos synchronization is easily realized in this coupling system. These attractors are disappeared by meeting tangent bifurcations G_1 and G_2. Thus there exist stable $C_\pm$ only in Region (F).

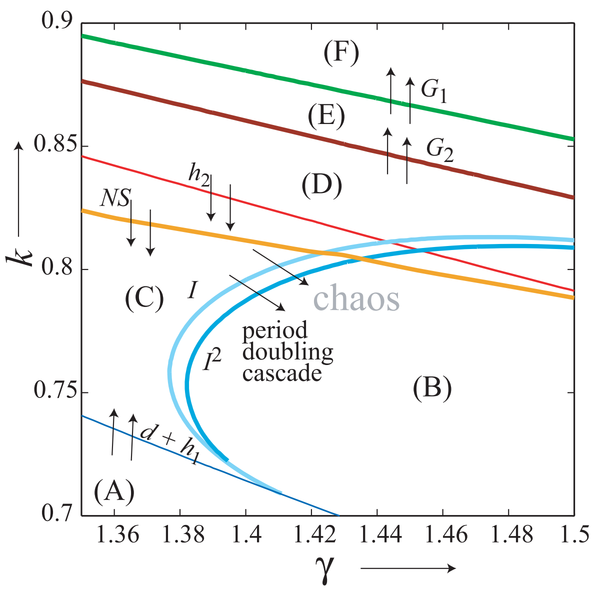


Figure 2: Bifurcation diagram in k-γ plane, $\delta = 0.1$

5. Bifurcation in k-δ plane

Figure 4 shows bifurcation diagram in k-δ plane with $\gamma = 1.45$. In this figure, G_0, G_1, I, I^2 correspond to those in Fig.

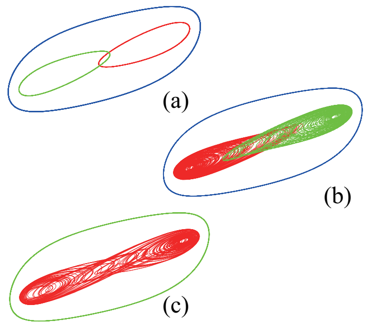


Figure 3: Typical synchronized attractors with $\delta = 0.1$. (a): three stable limit cycles observed in Region (C), $k = 0.814$, $\gamma = 1.425$, (b): two chaos attractors visualized different colors, $k = 0.8$, $\gamma = 1.446$, (c): fully developed chaos, $k = 0.798$, $\gamma = 1.452$.

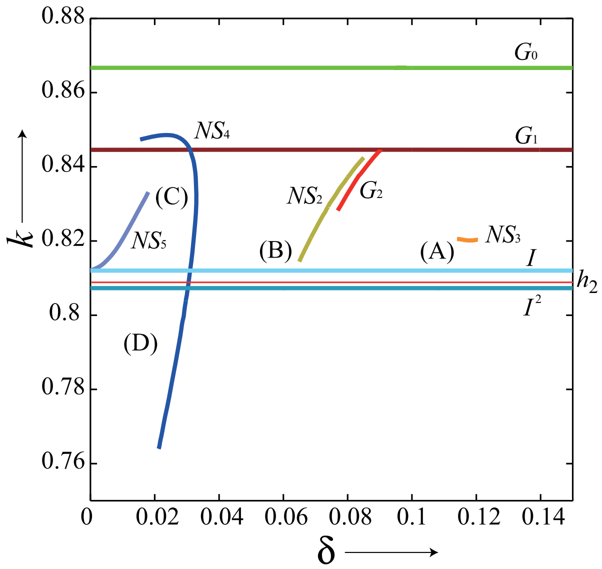


Figure 4: Bifurcation diagram in k-δ plane, $\gamma = 1.45$

1. Now the parameter δ declines from Region (A) to (B), then one of symmetric limit cycle becomes unstable, and the flow is absorbed by the big limit cycle. Moreover, as the parameter δ declined further from Region (B) to (C), three-phase synchronization of torus is obtained, see Fig. 5.

This three-phase synchronization torus becomes three-phase synchroned chaos, see Fig. 6. Furthermore, if the initial value is provided, another three-phase synchronized attractor is obtained; each oscillator has different amplitude, but retains three-phase synchronization, see Fig. 7. As the final example demonstlating a variation of this system, we show a partial anti-phase oscillation in Fig. 8. Two oscillators are synchronized with anti-phase, but they are no relation with the other oscillator.

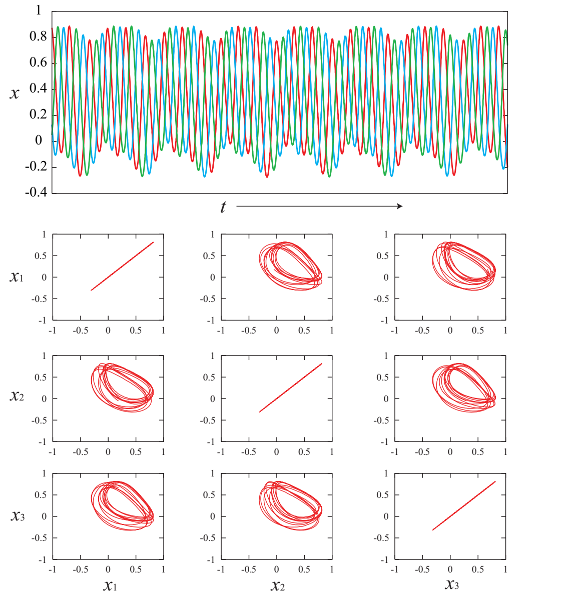


Figure 5: Three-phase synchronization of torus, $0 < t < 200$, $k = 0.822$, $\delta = 0.01$, $\gamma = 1.45$

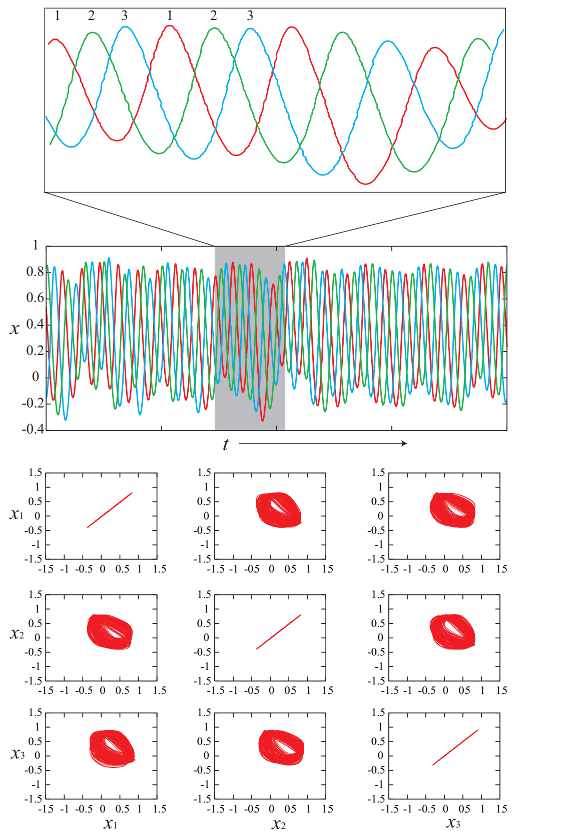


Figure 6: Three-phase synchronized chaos, $0 < t < 200$, $k = 0.818$, $\delta = 0.01$, $\gamma = 1.447$

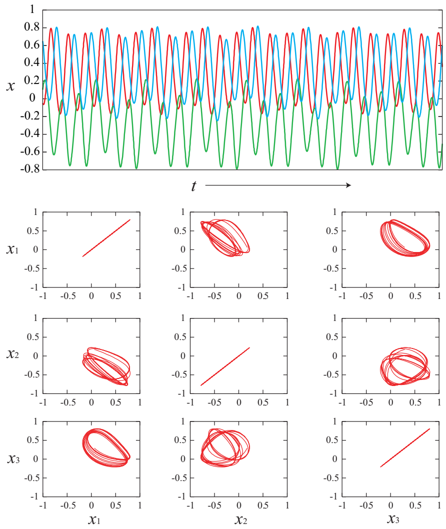


Figure 7: Three-phase synchronization with different attractors, $0 < t < 200$, $k = 0.82$, $\delta = 0.01$, $\gamma = 1.42$

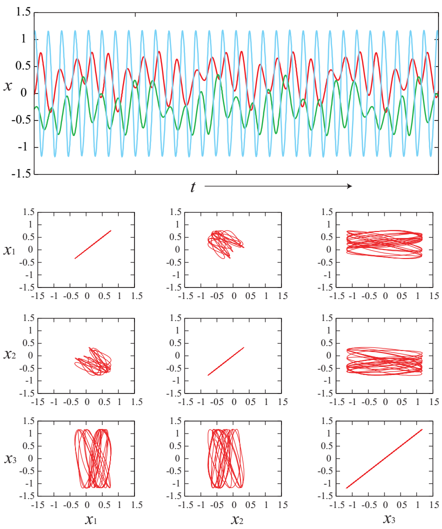


Figure 8: Anti-phase synchronization in two of the coupled circuit, $0 < t < 200$, $sk = 0.82$, $\delta = 0.01$, $\gamma = 1.42$

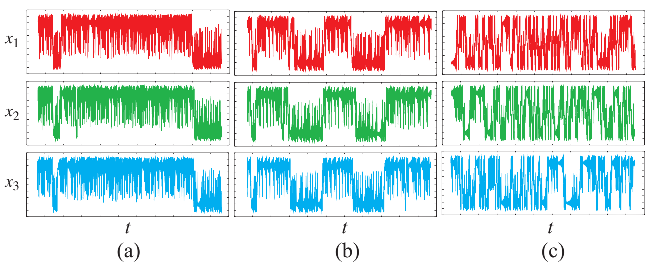


Figure 9: Self-switching synchronization and its breakdown, (a) $\delta = 0.007$, (b) $\delta = 0.004$, (c) $\delta = 0.001$, $|x_i| < 1$, $0 < t < 10000$.

6. Self-switching synchronization phenomena

Figure 9 (a) shows chaos attractor that observed in Region (D) in Figure 4. This attractor is similar to the one observed in Region (A). At this time, each coupled circuit is non synchronization(Fig. 9 (b)). However, The transition between attractors who adhere to one has synchronized(Fig. 9 (a)). This synchronization phenomena is collapses because it changes the parameter of δ.

7. Conclusions

In this study, we have showed several variations of synchronization phenomena in the extended BVP oscillators. Bifurcation diagrams reveals concrete parameter regions exhibiting completely in-phase chaos synchronization. As future works, we have to examine Lyapunov exponents for Fig. 5–9, and investigate attractors expected in the weak coupling case.

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