Phase-Inversion Waves in Simultaneously Existing Two Synchronization Modes of 2D Oscillator Network

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Abstract—We have been observing synchronization phenomena on coupled oscillators systems. We can observe special phenomena on lattice oscillators by using computer simulations. The special phenomena is the synchronization states in in-and-anti-phase synchronizations for vertical direction and in-phase synchronizations for horizontal direction. The synchronization states called two synchronization modes. In this paper, we investigate and analyze the phase-inversion waves on two synchronization modes on 2D oscillator networks. We separate the observation phenomena for 5 regions, and propagation mechanisms of the phase-inversion waves are analyzed.

1. Introduction

Synchronization phenomena can be observed in the biological bodies, the atomic world, the outer space, the mechanical systems, the electrical circuits, and so on. Especially, it is easy that synchronization phenomena are observed on coupled oscillators systems which are electrical circuits. Therefore, many phenomena have been reported by using electrical circuits[1]. When van der Pol oscillators are coupled by inductor as a ladder or as a 2D lattice, the all oscillators are an in-phase synchronization or an in-and-anti-phase synchronization. It is called the in-and-anti-phase synchronization that the in-phase synchronizations and the anti-phase synchronizations alternately exist. The in-and-anti-phase synchronization can be stable when phase states between edge oscillators and next oscillators are anti-phase synchronizations. On 2D oscillator network, we can observe that vertical synchronization states differ from horizontal synchronization states[2]. For example, if the vertical synchronization states are the in-phase synchronizations, the horizontal synchronization states are the in-and-anti-phase synchronizations. The synchronization states are called two synchronization modes.

In our previous study, we discovered a special wave motion that a phase state continuously propagates on a ladder oscillator networks or on 2D oscillator networks. Especially, we discovered a wave motion which switches phase states between adjacent oscillators and continuously exists. The wave motion is called phase-inversion waves[3].

In this study, we investigate and analyze the phase-inversion waves on two synchronization modes on 2D oscillator networks. We separate the observation phenomena for 5 regions. Furthermore, propagation mechanisms of the horizontal and the vertical phase-inversion waves are analyzed.

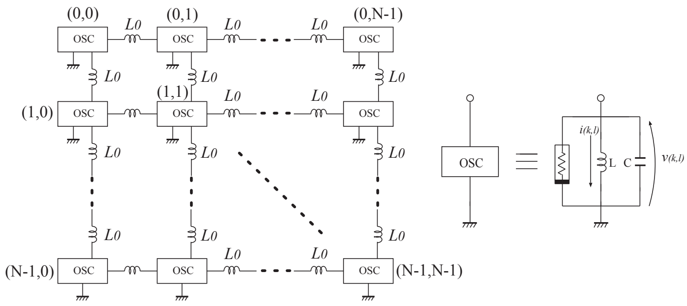

Figure 1: Circuit model.

2. Circuit model

The van der Pol oscillators are coupled by inductors L_0 as a 2D lattice (see Fig. 1) . The numbers of column and row of this system are assumed as "N" respectively. We name each oscillator OSC(k,l)($0 \leq k$ and $l \leq N-1$). A voltage of each oscillator is named $v_{(k,l)}$, and a current of a inductor in each oscillator is named $i_{(k,l)}$ (see Fig. 1) . An equation of the nonlinear negative resistor is shown as Eq. (1). Circuit equations are normalized by Eq. (2). The normalized circuit equations are shown as Eqs. (3)–(7). The α corresponds to a coupling parameter. The ε corresponds to a nonlinearity of each oscillator. This circuit is simulated by using the fourth order Runge-Kutta method and Eqs. (3)–(7).

$$i_r(v_{(k,l)}) = -g_1 v_{(k,l)} + g_3 v_{(k,l)}^3. \tag{1}$$

$$i_{(k,l)} = \sqrt{\frac{Cg_1}{3Lg_3}}x_{(k,l)}, \quad v_{(k,l)} = \sqrt{\frac{g_1}{3g_3}}y_{(k,l)},$$
$$t = \sqrt{LC}\tau, \quad \frac{d}{d\tau} = \text{“ } \cdot \text{ ”}, \alpha = \frac{L}{L_0}, \quad \varepsilon = g_1\sqrt{\frac{L}{C}}. \tag{2}$$

[Corner–top] (left:(a,b)=(0, 1), right:(a,b)=($N-1, N-2$).)

$$\frac{dx_{(0,a)}}{d\tau} = y_{(0,a)}, \tag{3}$$
$$\frac{dy_{(0,a)}}{d\tau} = -x_{(0,a)} + \alpha(x_{(0,b)} + x_{(1,a)} - 2x_{(0,a)}) + \varepsilon(y_{(0,a)} - \frac{1}{3}y_{(0,a)}^3).$$

[Corner–bottom] (left:(a,b)=(0, 1), right:(a,b)=($N-1, N-2$).)

$$\frac{dx_{(N,a)}}{d\tau} = y_{(N,a)}, \tag{4}$$

$$\frac{dy_{(N,a)}}{d\tau} = -x_{(N,a)} + \alpha(x_{(N-1,a)} + x_{(N,b)} - 2x_{(N,a)}) + \varepsilon(y_{(N,a)} - \tfrac{1}{3}y^3_{(N,a)}).$$

[Center] $(0 < k < N-1, 0 < l < N-1.)$

$$\frac{dx_{(k,l)}}{d\tau} = y_{(k,l)}, \tag{5}$$

$$\frac{dy_{(k,l)}}{d\tau} = -x_{(k,l)} + \alpha(x_{(k+1,l)} + x_{(k-1,l)} + x_{(k,l+1)} + x_{(k,l-1)} - 4x_{(k,l)}) + \varepsilon(y_{(k,l)} - \tfrac{1}{3}y^3_{(k,l)}).$$

[Edge]

(top:(a,b)=(0, 1),bottom:(a,b)=$(N-1, N-2)$,both:$0 < l < N-1$.)

$$\frac{dx_{(a,l)}}{d\tau} = y_{(a,l)}, \tag{6}$$

$$\frac{dy_{(a,l)}}{d\tau} = -x_{(a,l)} + \alpha(x_{(a,l-1)} + x_{(a,l+1)} + x_{(b,l)} - 3x_{(a,l)}) + \varepsilon(y_{(a,l)} - \tfrac{1}{3}y^3_{(a,l)}).$$

(left:(a,b)=(0, 1), right:(a,b)=$(N-1, N-2)$, both:$0 < k < N-1$.)

$$\frac{dx_{(k,a)}}{d\tau} = y_{(k,a)}, \tag{7}$$

$$\frac{dy_{(k,a)}}{d\tau} = -x_{(k,a)} + \alpha(x_{(k-1,a)} + x_{(k+1,a)} + x_{(k,b)} - 3x_{(k,a)}) + \varepsilon(y_{(k,a)} - \tfrac{1}{3}y^3_{(k,a)}).$$

3. Phase-invertion waves

Phase-inversion waves in the in-and-anti-phase synchronization for vertical direction and the in-phase synchronization for horizontal direction are shown in Figs. 2 and 3. The Figure 2 is observed when N is nine(an odd number). Signs of initial values are shown in Fig. 4. "X" expresses an attractor of each oscillator(current vs. voltage). "Y" expresses an itinerancy of phase difference by which sum of voltages of adjacent oscillators is shown along the time(sum of voltages vs. time). Black areas are almost the in-phase synchronization. White areas are almost the anti-phase synchronization. In each figure, we can observe phase-inversion waves which simultaneously propagate to horizontal direction in the in-phase synchronization and to vertical direction in the in-and-anti-phase synchronization. The Figure 3 is observed when N is ten(an even number). Signs of initial values are shown in Fig. 5. In this study, basic synchronization states of vertical direction are set as the in-and-anti-phase synchronization and synchronization states of horizontal direction are set as the in-phase synchronization.

3.1. Regions

We separate the observation phenomena for five regions. The coupling parameter α is changed from 0.010 to 0.30, every 0.010. The nonlinearity ε is changed from 0.010 to 0.50, every 0.010. A wave which doesn't disappear before 30000τ is assumed as the phase-inversion wave in this paper. The Figure 6 shows regions which the phase-inversion waves can be observed when N equals 9. The Figure 7 shows regions which the phase-inversion wave can be observed when N equals 10. The vertical and horizontal phase-inversion waves

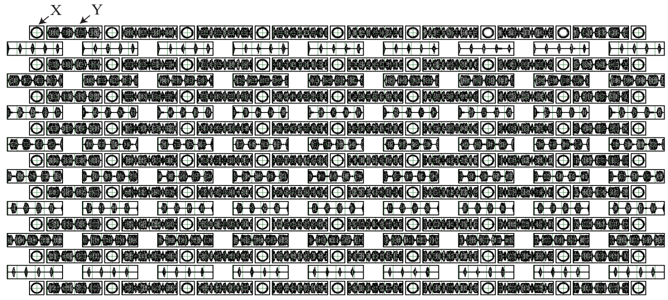

Figure 2: Computer simulation result of the phase-inversion waves in the in-and-anti-phase synchronization for vertical direction and the in-phase synchronization for horizontal direction on 9×9 2D oscillator.

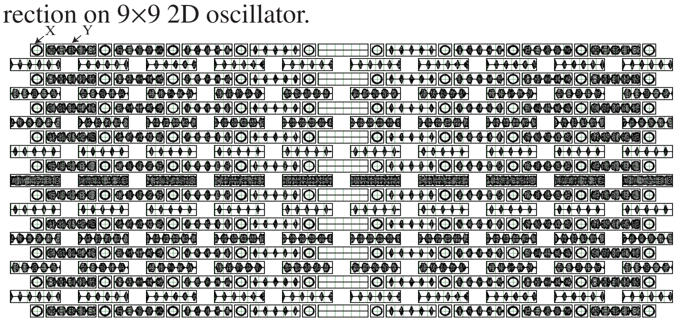

Figure 3: Computer simulation result of the phase-inversion waves in the in-and-anti-phase synchronization for vertical direction and the in-phase synchronization for horizontal direction on 10×10 2D oscillator.

can be observed in region(i)(see Figs. 6 and 7). The vertical phase-inversion waves can be observed and the horizontal phase-inversion waves can't be observed in region(ii)(see Fig. 6). The vertical phase-inversion waves and the complex phenomena for horizontal direction can be observed in region(iii)(see Figs. 6 and 7). The vertical phase-inversion waves and the synchronization phenomena without waves for horizontal direction can be observed in region(iv)(see Fig. 7). The complex phenomena for vertical direction and horizontal direction can be observed in region(v)(see Figs. 6 and 7).

3.2. Mechanisms

We can observe a phenomenon that vertical phase-inversion waves and horizontal phase-inversion waves propagate at the same time. We analyze the propagation mechanism of phase-inversion waves by using an instantaneous frequency of each oscillator and phase differences between adjacent oscillators. The coupling parameter is fixed as α=0.010, and nonlinearity is fixed as ε=0.150. The instantaneous frequency is defined as Eq. (8).

$$f_{(k,l)}(a) = \frac{1}{\tau_{(k,l)}(a) - \tau_{(k,l)}(a-1)}, \tag{8}$$

where "a" expresses the number of times of the voltage positive peak value. $\tau_{(k,l)}(a)$ means the time of OSC(k,l)(see Fig. 8). Similarly, $\tau_{(k+1,l)}(a)$ and $\tau_{(k,l+1)}(a)$ are defined. Some frequencies are observed in this phenomena. These frequencies are needed to consider the synchronization states for the vertical direction and the horizontal direction, because this system is 2-dimensional array. Five phase states have

+	−	−	−	−	−	−	−	−
+	−	−	−	−	−	−	−	−
−	+	+	+	+	+	+	+	+
−	+	+	+	+	+	+	+	+
+	−	−	−	−	−	−	−	−
+	−	−	−	−	−	−	−	−
−	+	+	+	+	+	+	+	+
−	+	+	+	+	+	+	+	+
+	−	−	−	−	−	−	−	−

Figure 4: Sign of initial value of each oscillator of Fig. 2.

+	+	+	+	+	−	−	−	−	−
+	+	+	+	+	−	−	−	−	−
−	−	−	−	−	+	+	+	+	+
−	−	−	−	−	+	+	+	+	+
+	+	+	+	+	−	−	−	−	−
+	+	+	+	+	−	−	−	−	−
−	−	−	−	−	+	+	+	+	+
−	−	−	−	−	+	+	+	+	+
+	+	+	+	+	−	−	−	−	−
+	+	+	+	+	−	−	−	−	−

Figure 5: Sign of initial value of each oscillator of Fig. 3.

to be considered for an oscillator of 2D lattice, because the in-phase synchronization and the anti-phase synchronization are exist in the same time. Therefore, five types synchronization states are observed in 2D lattice as follows:

1. fiiii: Four phase states are the in-phase synchronization.

2. fiiia: Three of four phase states are the in-phase synchronization. Another phase state is the anti-phase synchronization.
3. fiiaa: Two of four states are the in-phase synchronization. Other phase states are the anti-phase synchronization.
4. fiaaa: Three of four phase states are the anti-phase synchronization. Another phase state is the in-phase synchronization.
5. faaaa: Four phase states are the anti-phase synchronization.

The phase difference is calculated as follows. A phase difference between OSC(k,l) and OSC($k+1$, l) and a phase difference between OSC(k,l) and OSC(k, $l+1$) are obtained. The phase difference are assumed as $\Phi_{(k,l)(k+1,l)}(a)$ and $\Phi_{(k,l)(k,l+1)}(a)$, respectively. The $\Phi_{(k,l)(k+1,l)}(a)$ and $\Phi_{(k,l)(k,l+1)}(a)$ are obtained by Eq. (9) (see Fig. 8).

$$\begin{aligned}\Phi_{(k,l)(k+1,l)}(a) &= \frac{\tau_{(k,l)}(a)-\tau_{(k+1,l)}(a)}{\tau_{(k,l)}(a)-\tau_{(k,l)}(a-1)}\times 2\pi \text{ [rad]}\\ \Phi_{(k,l)(k,l+1)}(a) &= \frac{\tau_{(k,l)}(a)-\tau_{(k,l+1)}(a)}{\tau_{(k,l)}(a)-\tau_{(k,l)}(a-1)}\times 2\pi \text{ [rad].}\end{aligned} \quad (9)$$

Propagation mechanism of the phase-inversion wave is shown in Table 1(see Fig. 9), and the propagation mechanism of the horizontal phase-inversion wave is shown in Table 2(see Fig. 10)

4. Conclusion

We analyzed the phase-inversion waves on simultaneously existing two synchronization modes on 2D oscillator networks. We clarified regions of phenomena with the phase-inversion waves in the in-and-anti-phase synchronization for vertical direction and the in-phase synchronization for horizontal direction. Furthermore, the propagation mechanisms of the phase-inversion waves were analyzed.

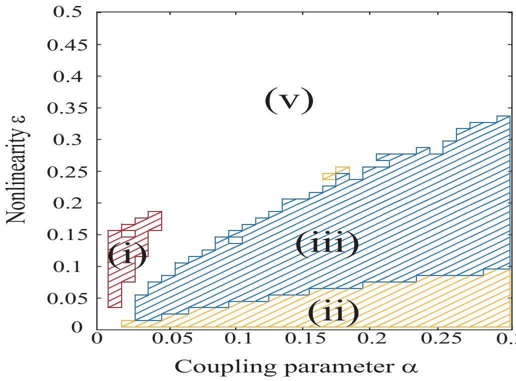

Figure 6: Regions of when N=9.

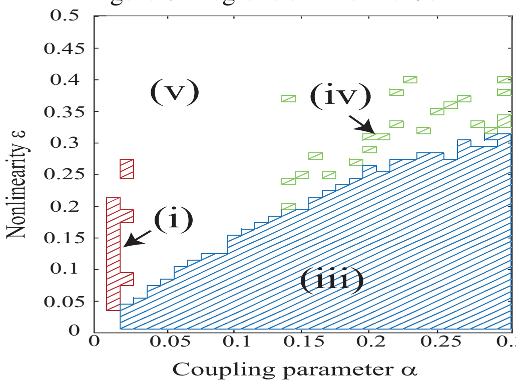

Figure 7: Regions of when N=10.

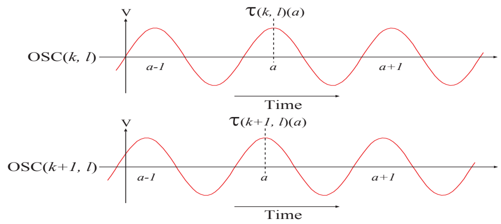

Figure 8: The calculation method of the instantaneous frequencies and the phase differences.

Acknowledgements

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Table 1: Propagation mechanism of a phase-inversion wave in a column (see Fig. 9).

no.	*Mechanism*
0	At first, $\Phi_{(3,3)(4,3)}$ is $-\pi$, $\Phi_{(4,3)(5,3)}$, $\Phi_{(4,2)(4,3)}$, $\Phi_{(4,3)(4,4)}$, $\Phi_{(5,2)(5,3)}$ and $\Phi_{(5,3)(5,4)}$ are 0, and $\Phi_{(5,3)(6,3)}$ is π. The vertical phase-inversion wave comes from 0th row to 3rd row in the in-and-anti-phase synchronization in each column.
1	$\Phi_{(3,3)(4,3)}$ starts to change from $-\pi$ toward -2π by the vertical phase-inversion wave.
2	$f_{(4,3)}$ starts to change from f_{iiia} to f_{iiii}, because $\Phi_{(3,3)(4,3)}$ is changing from $-\pi$ to -2π and $\Phi_{(4,2)(4,3)}$, $\Phi_{(4,3)(4,4)}$ and $\Phi_{(4,3)(5,3)}$ are 0.
3	$\Phi_{(4,3)(5,3)}$ starts to change from 0 to π, because $f_{(4,3)}$ is changing from f_{iiia} to f_{iiii}, and $f_{(5,3)}$ is not changed from f_{iiia} yet. $\Phi_{(4,2)(4,3)}$ and $\Phi_{(4,3)(4,4)}$ are 0, because $f_{(4,2)}$, $f_{(4,3)}$, and $f_{(4,4)}$ are changing from f_{iiia} to f_{iiii} at same time by each vertical phase-inversion wave.
4	$f_{(5,3)}$ starts to change from f_{iiia} toward f_{iiaa}, because $\Phi_{(4,3)(5,3)}$ is changing from 0 to π, $\Phi_{(5,2)(5,3)}$ and $\Phi_{(5,3)(5,4)}$ are 0, and $\Phi_{(5,3)(6,3)}$ is π.
5	$\Phi_{(5,3)(6,3)}$ starts to change from π toward 0, because $f_{(5,3)}$ is changing from f_{iiia} to f_{iiaa}.
6	$f_{(4,3)}$ starts to change toward f_{iiia} again without arriving at f_{iiii}, because $\Phi_{(3,3)(4,3)}$ is changing to 2π, $\Phi_{(4,3)(5,3)}$ is changing to π, and $\Phi_{(4,2)(4,3)}$ and $\Phi_{(4,3)(4,4)}$ is 0. So, three of four phase states around OSC(4,3) are changing toward the in-phase synchronization state, and another phase state is changing toward anti-phase synchronization.
7	$f_{(5,3)}$ starts to change toward f_{iiia} again without arriving at f_{iiaa}, because $\Phi_{(4,3)(5,3)}$ is changing to π, $\Phi_{(5,3)(6,3)}$ is changing to 0, and $\Phi_{(5,2)(5,3)}$ and $\Phi_{(5,3)(5,4)}$ is 0.
8	$\Phi_{(3,3)(4,3)}$ arrives at -2π.
9	$f_{(4,3)}$ arrives at f_{iiia} again.
10	$\Phi_{(4,3)(5,3)}$ arrives at π.
11	$f_{(5,3)}$ arrives at f_{iiia} again.
12	$\Phi_{(5,3)(6,3)}$ arrives at 0.
The vertical phase-inversion wave propagate by this mechanism.	

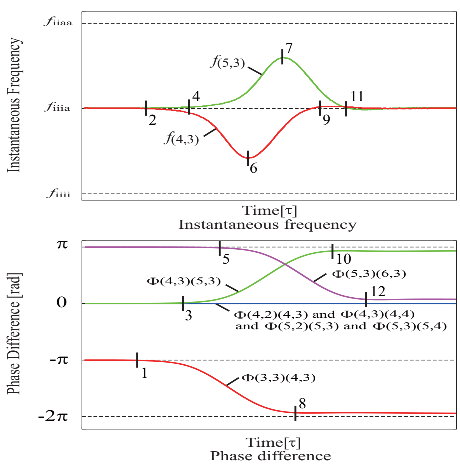

Figure 9: Transitions of frequencies and phase differences by propagation of a phase-inversion wave in columns.

Table 2: Propagation mechanism of a phase-inversion wave in a row (see Fig. 10).

no.	*Mechanism*
0	$\Phi_{(2,13)(2,14)}$, $\Phi_{(2,14)(2,15)}$, $\Phi_{(2,15)(2,16)}$, $\Phi_{(1,14)(2,14)}$ and $\Phi_{(1,15)(2,15)}$ are 0. $\Phi_{(2,14)(3,14)}$ and $\Phi_{(2,15)(3,15)}$ are π. The horizontal phase-inversion wave comes from 11th column to 13th column in the in-phase synchronization in each row.
1	$\Phi_{(2,13)(2,14)}$ starts to change from 0 toward $-\pi$ by the horizontal phase-inversion wave.
2	$f_{(2,14)}$ starts to change from f_{iiia} toward f_{iiaa}, because $\Phi_{(2,13)(2,14)}$ is changing from 0 to $-\pi$, $\Phi_{(1,14)(2,14)}$ and $\Phi_{(2,14)(2,15)}$ are 0 and $\Phi_{(2,14)(3,14)}$ is π. $\Phi_{(1,14)(2,14)}$ and $\Phi_{(2,14)(3,14)}$ don't change, because $f_{(1,14)}$, $f_{(2,14)}$, and $f_{(3,14)}$ are similarly changed by each horizontal phase-inversion wave.
3	$\Phi_{(2,14)(2,15)}$ starts change from 0 toward $-\pi$, because $f_{(2,14)}$ is changing from f_{iiia} to f_{iiaa}.
4	$f_{(2,14)}$ continuously changes toward f_{iaaa}, because $\Phi_{(2,13)(2,14)}$ and $\Phi_{(2,14)(2,15)}$ are changing from 0 to $-\pi$, $\Phi_{(1,14)(2,14)}$ is 0 and $\Phi_{(2,14)(3,14)}$ is π.
5	$f_{(2,15)}$ starts to change from f_{iiia} toward f_{iiaa}, because $\Phi_{(2,14)(2,15)}$ is changing from 0 to $-\pi$, $\Phi_{(2,15)(2,16)}$ and $\Phi_{(1,15)(2,15)}$ are 0 and $\Phi_{(2,15)(3,15)}$ is π.
6	$\Phi_{(2,15)(2,16)}$ starts to change from 0 toward π, because $f_{(2,15)}$ is changing from f_{iiia} to f_{iiaa}.
7	$f_{(2,15)}$ continuously changes toward f_{iaaa}, because $\Phi_{(2,14)(2,15)}$ and $\Phi_{(2,15)(2,16)}$ are changing from 0 to π, $\Phi_{(1,15)(2,15)}$ is 0 and $\Phi_{(2,15)(3,15)}$ is π.
8	$\Phi_{(2,13)(2,14)}$ arrives at $-\pi$.
9	$f_{(2,14)}$ arrives at f_{iaaa}.
10	$\Phi_{(2,14)(2,15)}$ arrives at $-\pi$.
11	$f_{(2,15)}$ arrives at f_{iaaa}.
12	$\Phi_{(2,15)(2,16)}$ arrives at $-\pi$.
The horizontal phase-inversion wave propagate by this mechanism.	

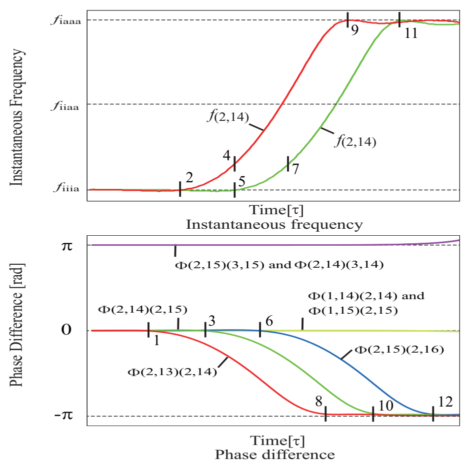

Figure 10: Transitions of frequencies and phase differences by propagation of a phase-inversion wave in rows.