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# Global Asymptotic Stability Analysis of Nonlinear Circuits for Solving the Maximum Flow Problem

Norikazu Takahashi

Department of Informatics, Kyushu University
744 Motooka, Nishi-ku, Fukuoka 819-0395 Japan
Email: norikazu@inf.kyushu-u.ac.jp

**Abstract**—Global asymptotic stability of a nonlinear circuit for solving the maximum flow problem, which was first proposed by Sato *et al.*, is studied in this paper. The circuit consists of two independent DC voltage sources, capacitors and nonlinear resistors. It is proved rigorously that the circuit has a unique equilibrium point which is globally asymptotically stable. From the viewpoint of dynamical systems, the circuit is a cooperative system, and thus some fundamental results concerning the convergence property of cooperative systems play important roles.

## 1. Introduction

From the mid-eighties, there have been many attempts to solve optimization problems by using nonlinear circuits [1, 2, 3, 4]. One of the most important examples is the Hopfield neural network [1]. This continuous-time recurrent neural network model, which can be implemented by a nonlinear circuit, is a powerful tool for finding an approximate solution of the traveling salesman problem. Also, some authors proposed to use SPICE, the most widely used circuit simulator, for solving constrained optimization problems [3, 4]

Recently, Sato *et al.* [5] proposed a nonlinear circuit[1] for solving the maximum flow problem [6, 7, 8]. The circuit consists of two independent DC voltage sources, capacitors and nonlinear resistors. They performed a number of computer simulations and observed that the circuit always converges to an equilibrium point that corresponds to the maximum flow. However, the convergence property of the circuit has not been completely understood so far.

In this paper, it is proved under certain mild conditions that the nonlinear circuit proposed by Sato *et al.* [5] has a unique equilibrium point which is globally asymptotically stable. From the viewpoint of dynamical systems, the circuit belongs to an important class of dynamical systems called the cooperative system [9, 10]. Thus the proof given in this paper is based on a fundamental result [11] concerning the global stability of cooperative systems. First, the boundedness of state trajectories is proved. Second, it is proved that every equilibrium point is locally asymptotically stable. Third, the uniqueness of the equilibrium point is proved by making use of the Brouwer degree [12, 13]. Finally, it is proved that the unique equilibrium point is globally asymptotically stable.

## 2. Maximum Flow Problem

First of all, we briefly review the maximum flow problem. Let $G = (V, E)$ be a simple directed graph where $V = \{v_0, v_1, \dots, v_{n+1}\}$ is the set of vertices and $E = \{e_1, e_2, \dots, e_m\}$ is the set of edges. An edge $e_k \in E$ directed from $v_i \in V$ to $v_j \in V$ is denoted by $e_k = (v_i, v_j)$. The set $V$ contains two distinguished vertices: the source $v_0$ and the sink $v_{n+1}$. The source $v_0$ is the vertex such that $E$ contains no edge directed to $v_0$. On the other hand, the sink $v_{n+1}$ is the vertex such that $E$ contains no edge directed from $v_{n+1}$. Throughout this paper, we assume:

**Assumption 1** For each vertex $v_i \in V \setminus \{v_0, v_{n+1}\}$, there is at least one directed path from $v_0$ to $v_i$ and there is at least one directed path from $v_i$ to $v_{n+1}$.

Let $c : E \to \mathbb{R}_+$ be a capacity function where $\mathbb{R}_+$ is the set of positive numbers. The capacity of an edge $(v_i, v_j)$ is denoted by $c(v_i, v_j)$. A flow on the graph $G$ is a function $f : E \to \mathbb{R}$ satisfying the following conditions:

$$0 \le f(v_i, v_j) \le c(v_i, v_j), \quad \forall (v_i, v_j) \in E$$

$$\sum_{j:(v_j,v_i)\in E} f(v_j, v_i) = \sum_{j:(v_i,v_j)\in E} f(v_i, v_j), \quad i = 1, 2, \dots, n$$

The maximum flow problem is to find a flow $f$ which maximizes

$$|f| \triangleq \sum_{j:(v_0,v_j)\in E} f(v_0, v_j).$$

An example of simple directed graphs is shown in Fig. 1 where $v_0$ and $v_5$ are the source and the sink, respectively, and numbers beside edges represent the capacity.

## 3. Nonlinear Circuits for Solving Maximum Flow Problems

Sato *et al.* [5] proposed a nonlinear circuit for solving the maximum flow problem. Their circuit consists of two independent DC voltage sources, $n$ capacitors and $m$ nonlinear resistors (see Fig. 2). Nodes of the circuit correspond

[1] In their paper [5], the circuit is called the maximum-flow neural network.

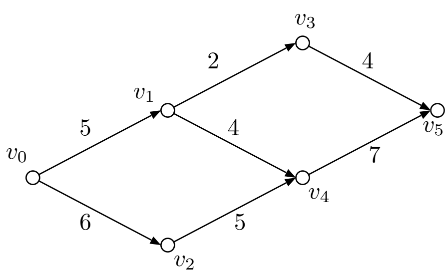


Figure 1: A simple directed graph [8].

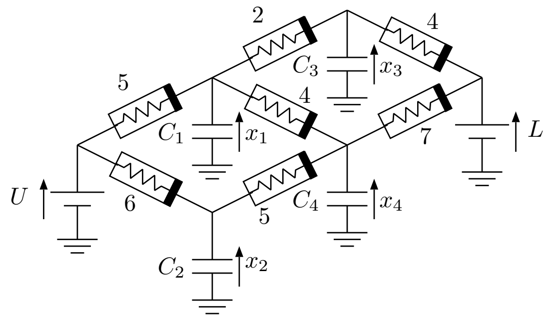


Figure 2: A nonlinear circuit for solving the maximum flow problem on the graph shown in Fig. 1.

to vertices of a given graph. The voltage-current characteristic of a nonlinear resistor shown in Fig. 3 is given by

$$i = A\sigma(v)$$

where $\sigma(\cdot)$ is a nonlinear function defined by

$$\sigma(y) = \frac{1}{1 + \exp(-y)}\,.$$

By taking the voltages of $n$ capacitors as variables, we obtain the following set of differential equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{1}{C_i}\Big[A_{0i}\sigma(U - x_i) + \sum_{j=1}^{n}\{A_{ji}\sigma(x_j - x_i) - A_{ij}\sigma(x_i - x_j)\} - A_{i,n+1}\sigma(x_i - L)\Big] \triangleq F_i(x), \quad i = 1, 2, \ldots, n \qquad (1)$$

where $x_i$'s are variables, $U$ and $L$ are constants that satisfy the inequality

$$L < U,$$

$C_i$'s are positive constants, $A_{ij}$'s are nonnegative constants defined by

$$A_{ij} = \begin{cases} c(v_i, v_j), & \text{if } (v_i, v_j) \in E \\ 0, & \text{otherwise}\,. \end{cases}$$

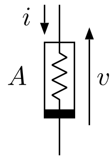


Figure 3: The $v$-$i$ characteristic of the nonlinear resistor.

Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ be a solution of the differential equation (1). Then

$$\lim_{t\to\infty} A_{ij}\sigma(x_i(t) - x_j(t)), \quad i, j = 0, 1, \ldots, n$$

correspond to an approximate solution of the maximum flow problem. Therefore, the circuit must converge to an equilibrium point for any initial condition in order to work properly as a maximum flow problem solver.

## 4. Global Asymptotic Stability Analysis

### 4.1. Boundedness of Solutions

**Lemma 1** Let $\Omega = [L, U]^n \subset \mathbb{R}^n$. Every solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ of the system (1) belongs to $\Omega$ for all $t \geq 0$ if $x(0) \in \Omega$.

*Proof.* Let $x$ be any point in $\partial\Omega$ where $\partial\Omega$ represents the boundary of $\Omega$. If $x_i = L$ then $\sigma(U - x_i)$, $\sigma(x_j - x_i)$ $(j = 1, 2, \ldots, n)$, $-\sigma(x_i - x_j)$ $(j = 1, 2, \ldots, n)$, and $-\sigma(x_i - L)$ are all nonnegative because $\sigma$ is a monotone increasing function. Hence $F_i(x)$ is nonnegative. Similarly we can show that $F_i(x)$ is nonpositive if $x_i = U$. Therefore, any solution $x(t)$ such that $x(0) \in \Omega$ cannot leave $\Omega$. In other words, $\Omega$ is a positively invariant set for the system (1). □

### 4.2. Local Asymptotic Stability of an Equilibrium Point

**Lemma 2** If $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}} \in \Omega = [L, U]^n$ is an equilibrium point of the system (1) then $x^* \in \operatorname{int}\Omega$ and is locally asymptotically stable.

*Proof.* Suppose that the system (1) has an equilibrium point $x^* \in \partial\Omega$. Then there exists at least one $i$ such that $x_i^* = L$ or $x_i^* = U$. In the case where $x_i^* = L$, the value of $x_j^*$ must be $L$ for all $j$ such that $(v_j, v_i) \in E$ because otherwise $F_i(x^*)$ is positive which contradicts the assumption that $x^*$ is an equilibrium point. For the same reason, the value of $x_k^*$ must be $L$ for all $k$ such that $(v_k, v_j) \in E$ and $(v_j, v_i) \in E$ for some $j$. By repeating this discussion, we reach the conclusion that there exists an integer $l$ such that $(v_0, v_l) \in E$ and the value of $x_l^*$ must be $L$. However, this implies that $F_l(x^*)$ is positive which leads to a contradiction. In the case where $x_i^* = U$, we can show in the same way as above that there exists an integer $l$ such that $(v_l, v_{n+1}) \in E$ and the value of $x_l^*$ must be $U$, which leads to a contradiction. This completes the proof of the first statement.

For the second statement, let us consider the Jacobian matrix $J \in \mathbb{R}^{n\times n}$ of the vector field $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^{\mathrm{T}}$ at an equilibrium point $x^*$. The $(i, j)$ element of $J$ is given by

$$J_{ij} = \begin{cases} [-A_{0i}\sigma'(U - x_i^*) - \sum_{k=1}^{n}\{A_{ki}\sigma'(x_k^* - x_i^*) \\ \quad +A_{ik}\sigma'(x_i^* - x_k^*)\} - A_{i,n+1}\sigma'(x_i^* - L)]/C_i, & i = j \\ [A_{ji}\sigma'(x_j^* - x_i^*) + A_{ij}\sigma'(x_i^* - x_j^*)]/C_i, & i \neq j \end{cases} \tag{2}$$

Note that the nonnegative constant $A_{ij}$ is positive if and only if $(v_i, v_j) \in E$ and that $\sigma$ is a monotone increasing function. From these facts and Assumption 1, we see that every diagonal element of $J$ is negative, every off-diagonal element of $J$ is nonnegative, and $J$ is irreducible (for the definition of the irreducible matrix, see [14] for example). Also, it is easily seen from (2) that $J$ satisfies

$$|J_{ii}| \geq \sum_{j=1, j\neq i}^{n} |J_{ij}|, \quad i = 1, 2, \ldots, n.$$

In particular,

$$|J_{kk}| > \sum_{j=1, j\neq k}^{n} |J_{kj}|$$

holds for all $k$ such that $(v_0, v_k) \in E$ or $(v_k, v_{n+1}) \in E$. Hence $J$ is irreducibly diagonally dominant [14]. It is well known that if a square matrix is irreducibly diagonally dominant then it is nonsingular and if, in addition, its diagonal elements are negative then every eigenvalue has negative real part [14, Theorem 4.9]. Therefore, $J$ is nonsingular and every eigenvalue has negative real part, which means that the equilibrium point $x^*$ is locally asymptotically stable. □

### 4.3. Uniqueness of Equilibrium Point

**Lemma 3** The system (1) has a unique equilibrium point in $\Omega = [L, U]^n$.

*Proof.* We first consider the system of linear differential equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{1}{C_i}\Big[A_{0i}(U - x_i) + \sum_{j=1}^{n}\{A_{ji}(x_j - x_i) - A_{ij}(x_i - x_j)\} - A_{i,n+1}(x_i - L)\Big] \triangleq G_i(x), \quad i = 1, 2, \ldots, n. \tag{3}$$

Let $C = \mathrm{diag}(C_1, C_2, \ldots, C_n) \in \mathbb{R}^{n\times n}$. Let a constant matrix $K = (K_{ij}) \in \mathbb{R}^{n\times n}$ and a constant vector $b = (b_1, b_2, \ldots, b_n)^{\mathrm{T}} \in \mathbb{R}^n$ be defined as

$$K_{ij} = \begin{cases} -A_{0i} - \sum_{k=1}^{n}(A_{ki} + A_{ik}) - A_{i,n+1}, & \text{if } i = j \\ A_{ji} + A_{ij}, & \text{if } i \neq j \end{cases}$$

$$b_i = A_{0i}U + A_{i,n+1}L.$$

Then (3) can be rewritten in a matrix form as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = C^{-1}(Kx + b) \triangleq G(x). \tag{4}$$

By applying the same argument as in the proof of the second part of Lemma 2 to $C^{-1}K$, we can show that $C^{-1}K$ is nonsingular and every eigenvalue of $C^{-1}K$ has negative real part. In particular, $K$ is nonsingular and every eigenvalue of $K$ is a negative real number because $K$ is symmetric. Hence the system (3) or (4) has a unique equilibrium point $\hat{x} = -K^{-1}b \in \mathbb{R}^n$ which is locally asymptotically stable. Moreover, we can show that $\hat{x} \in \mathrm{int}\,\Omega$ as follows. Suppose that $\hat{x} \notin \mathrm{int}\,\Omega$. Let $i_1 = \mathrm{argmin}_{1\leq i\leq n}\,\hat{x}_i$ and $i_2 = \mathrm{argmax}_{1\leq i\leq n}\,\hat{x}_i$. Then, at least one of two inequalities: $\hat{x}_{i_1} \leq L$ and $\hat{x}_{i_2} \geq U$ holds. In the former case, by applying the same argument as in the proof of the first part of Lemma 2, we reach the conclusion that there must exist an integer $i$ such that $(v_0, v_i) \in E$ and $\hat{x}_i = L$. However, this leads to a contradiction. In the latter case, we reach the conclusion in the same way that there must exist an integer $i$ such that $(v_i, v_{n+1}) \in E$ and $\hat{x}_i = U$. However, this also leads to a contradiction. Therefore we can conclude that $\hat{x} \in \mathrm{int}\,\Omega$.

The Brouwer degree [12, 13] of the vector field $G(x) = (G_1(x), G_2(x), \ldots, G_n(x))^{\mathrm{T}}$ with respect to $\mathrm{int}\,\Omega$ and value 0, which is denoted by $d(G, \mathrm{int}\,\Omega, 0)$[2], satisfies

$$d(G, \mathrm{int}\,\Omega, 0) = \mathrm{sgn}(|C^{-1}K|) = \mathrm{sgn}(|K|) = (-1)^n$$

where the last equality follows from the fact that $n$ eigenvalues of $K$ are real and negative. Let $H(x, s)$ be defined by

$$H(x, s) \triangleq sF(x) + (1 - s)G(x).$$

It is obvious that $H(x, s)$ is continuous on $\Omega \times [0, 1]$, $H(x, 0) = G(x)$ and $H(x, 1) = F(x)$. Moreover, we claim that $H(x, s) \neq 0$ for all $x \in \partial\,\Omega$ and all $t \in (0, 1)$. To see this, suppose that $H(\tilde{x}, \tilde{s}) = 0$ for some $\tilde{x} \in \partial\,\Omega$ and $\tilde{s} \in (0, 1)$. Then we have

$$F(\tilde{x}) = -\frac{1 - \tilde{s}}{\tilde{s}}G(\tilde{x}). \tag{5}$$

However, since $G_i(\tilde{x}) > 0$ for all $i$ such that $\tilde{x}_i = L$ and $G_i(\tilde{x}) < 0$ for all $i$ such that $\tilde{x}_i = U$, (5) implies that there exists at least one $i$ such that one of two conditions:

1) $\tilde{x}_i = L$ and $F_i(\tilde{x}) < 0$

2) $\tilde{x}_i = U$ and $F_i(\tilde{x}) > 0$

holds, which contradicts Lemma 1. Therefore, two vector fields $F$ and $G$ are homotopic. Since the Brouwer degree is homotopy invariant, we have

$$d(F, \mathrm{int}\,\Omega, 0) = (-1)^n. \tag{6}$$

[2] This paper employs the notation used in [12]. In Reference [13], the Brouwer degree is denoted by $\deg(G, 0, \mathrm{int}\,\Omega)$.

On the other hand, it follows from the definition of the Brouwer degree and Lemma 2 that

$$d(F, \mathrm{int}\,\Omega, 0) = m \times (-1)^n \tag{7}$$

where $m$ is the number of equilibrium points of the system (1) in $\mathrm{int}\,\Omega$. From (6) and (7), we can conclude that $m$ must be one, that is, the system (1) has a unique equilibrium point in $\Omega$. □

### 4.4. Global Asymptotic Stability of the Unique Equilibrium Point

**Theorem 1** The system (1) has a unique equilibrium point in $\Omega = [L, U]^n$ which is globally asymptotically stable.

*Proof.* The system (1) is a $C^1$ cooperative system [9, 10, 11] on $\Omega$ because

$$\frac{\partial F_i(x)}{\partial x_j} \geq 0$$

holds for all $i \neq j$ and all $x \in \Omega$. It is known that a $C^1$ cooperative system in a closed box $X \subset \mathbb{R}^n$ has a globally asymptotically stable equilibrium point if and only if two conditions:

1) The system has a unique equilibrium point in $X$.

2) Every forward semi-orbit has compact closure in $X$.

hold [11, Theorem C]. Since we have already seen in Lemmas 1 through 4 that the system (1) satisfies these conditions with $X = \Omega$, it has a unique equilibrium point which is globally asymptotically stable. □

## 5. Conclusion

It was proved in this paper that the nonlinear circuit for solving the maximum flow problem has a unique equilibrium point which is globally asymptotically stable. The main result of this paper is restricted to nonlinear resistors such that the voltage-current characteristic is sigmoidal and cannot be applied to piecewise-linear resistors. Extension of the result to the nonlinear circuit with piecewise-linear resistors is a future problem. Another future problem is to clear the relation between the unique equilibrium point of the circuit and the maximum flow of the graph.


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