

Arnold tongues of 2-D Piecewise Constant Driven Oscillator

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Abstract—In this study, we consider bifurcation phenomena of a two-dimensional piecewise-constant driven oscillator. The system exhibits phase-locking phenomena in a region in the space of parameters. The region is known as Arnold tongues. We propose a novel method in order to analyze piecewise-constant systems with external force. By using the method, we show theoretical result of stability and bifurcation of the piecewise-constant system driven by an external force, especially, we strictly derive boundaries of Arnold tongues.

1. Introduction

In biology and engineering systems, forced synchronization phenomena are often observed [1][2]. It is well known that phase-locked regions called Arnold tongues are often observed in forced synchronization [3]. It is important to analyze Arnold tongues for understanding the forced synchronizations.

In the past, piecewise-linear systems that can obtain explicit solutions in each piecewise-linear regions were often used to consider the synchronization phenomena [4][5]. However, when the solution of the each region connects another one, it is necessary to solve implicit equations. In other words, numerical computations are needed.

For such problems, Tsubone *et al.* have proposed piecewise-constant systems [6]. Piecewise-constant systems governed by piecewise-constant vector fields have straight orbits. Piecewise solutions can be not only determined as linear equations, the connections of solution can be determined as explicit equation. Therefore, it is a good example of analyzing various phenomena. In addition, rigorous analysis method have been proposed [7]. Analysis of piecewise-constant systems with external force are also reported [8]. However, it is not sufficient for consideration of two or more dimensions non-autonomous systems.

In this paper, we consider bifurcation phenomena that occur in a 2-D piecewise-constant driven oscillator. We propose a novel method in order to analyze piecewise-constant systems with external force. By using the method, we show theoretical result of stability and bifurcation of the piecewise-constant system with external force. Furthermore, we strictly derive boundaries of Arnold tongues.

2. 2-D Piecewise-constant Driven Oscillator

Figure 1 shows a circuit schematic diagram of a 2-D piecewise-constant driven system. The circuit consists of two capacitors, one VCCS having a signam characteristic as shown in Fig. 2(a), two VCCSs having a hysteresis characteristic as shown in Fig. 2(b), and an independent current source.

The circuit dynamic is represented as follows.

$$\begin{cases} C_1 \frac{dv_1}{dt} &= I_1 \cdot H(v_1) + I_3 \cdot \text{sgn}(v_2), \\ C_2 \frac{dv_2}{dt} &= I_2 \cdot H(v_1) + I_4 \cdot B(T, t), \end{cases} \quad (1)$$

where $B(T, t)$ is a current source as shown in Fig. 3, $\text{sgn}(\cdot)$ and $H(\cdot)$ are voltage control current sources (VCCSs) that have characteristic as shown in Fig. 2(a) and 2(b), respectively.

$$B(T, t) = \begin{cases} 1, & \text{for } nT \leq t < \frac{(2n+1)T}{2}, \\ -1, & \text{for } \frac{(2n+1)T}{2} \leq t < (n+1)T. \end{cases} \quad (2)$$

In order to realize oscillation behavior, we consider following conditions.

$$I_2 = -I_3, I_1 \cdot I_2 < 0. \quad (3)$$

Here, by using the following normalized variables and parameters

$$\begin{aligned} \tau &= \frac{I_2}{C_1 v_{th}} t, \quad x = \frac{1}{I_{th}} v_1, \quad y = \frac{C_2}{C_1 v_{th}} v_2, \\ \alpha &= -\frac{I_1}{I_2}, \quad \beta = \frac{I_4}{I_2}, \quad T' = \frac{I_2}{C_1 v_{th}} T, \end{aligned} \quad (4)$$

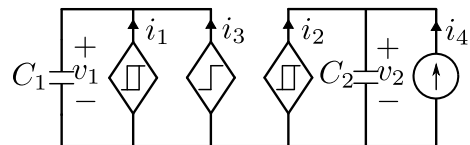


Figure 1: Circuit model.

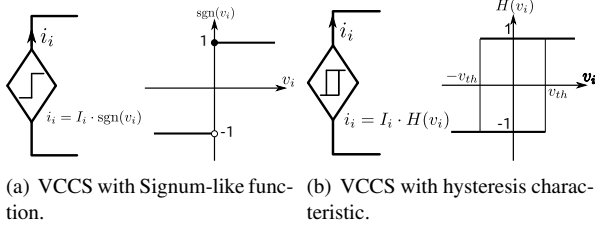


Figure 2: Symbols and nonlinear characteristics of VCCSs.

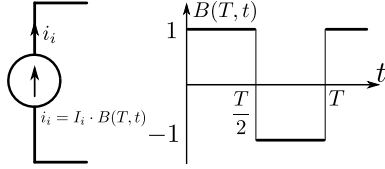


Figure 3: Independence current source.

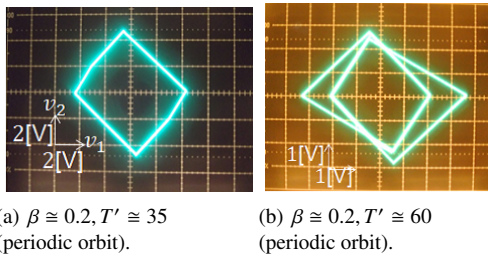
we can rewrite the circuit dynamics by following normalized equations,

$$\begin{cases} \dot{x} = -\alpha \cdot h(x) - \text{sgn}(y), \\ \dot{y} = h(x) + \beta \cdot B(T', \tau), \end{cases} \quad (5)$$

where “ $\dot{\cdot}$ ” denote differentiation by normalized time τ and h is a normalized hysteresis. h is switched from 1 to -1 if x reaches to the threshold -1 and h is switched from -1 to 1 if x reaches to 1. Here, we assume a following parameter conditions.

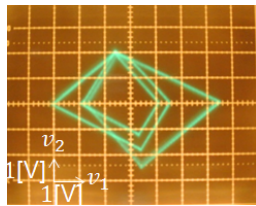
$$0 < \alpha < 1 \text{ and } 0 < \beta < 1. \quad (6)$$

The condition (6) guarantees oscillatory dynamics. Typical attractors are shown in Fig. 4.



(a) $\beta \cong 0.2, T' \cong 35$
(periodic orbit).

(b) $\beta \cong 0.2, T' \cong 60$
(periodic orbit).



(c) $\beta \cong 0.37, T' \cong 50$ (chaos).

Figure 4: Typical attractors ($\alpha \cong 0.2$).

3. Analysis Method

In order to analysis, we derive a novel calculation algorithm for rigorous solutions. In our previous work [7], the basic algorithm depends on 2-D mapping procedure related to state variables. However, the algorithm is not suitable for non-autonomous system, because it can not manage enforced switching depending on external force. So, we propose a novel algorithm based on 3-D mapping with time variable τ .

Step 1.

We set the initial state \mathbf{x}_0 and get the dependent variable l_0 .

$$l_k = \frac{-B(T', \tau) + 1}{2} \cdot 2^2 + \frac{-h(x) + 1}{2} \cdot 2^1 + \frac{-\text{sgn}(y) + 1}{2} \cdot 2^0, \quad (7)$$

where $k = 0$ and x_k, y_k and τ_k denote the elements of \mathbf{x}_k , that is, $\mathbf{x}_k = {}^t(x_k, y_k, \tau_k)$.

Step 2.

We calculate a time τ_k until the l_k switches to l_{k+1} . Assuming a trajectory started from \mathbf{x}_k arrives one of thresholds $D_x(l_k), D_y(l_k)$ and $D_\tau(l_k)$, each arrival times τ_x, τ_y and τ_τ are given by

$$\tau_x = \frac{D_x(l_k) - x_k}{a_x(l_k)}, \quad \tau_y = \frac{D_y(l_k) - y_k}{a_y(l_k)}, \quad (8)$$

$$\tau_\tau = D_\tau(l_k) - \tau_k \text{ mod } T', \quad (9)$$

where $a_x(l_k)$ and $a_y(l_k)$ denote the elements of $\mathbf{a}(l_k)$, that is, $\mathbf{a}(l_k) = {}^t(a_x(l_k), a_y(l_k), a_\tau(l_k))$ and $\mathbf{a}(l_k)$ is obtained from Table 1. The actual arrival time τ_k that means the switching time of l_k is given by the minimum of τ_x, τ_y and τ_τ omitting zero and negative.

$$\tau_k = \min\{\{\tau_x, \tau_y, \tau_\tau\} \cap \{\xi \in \mathbf{R} | \xi > 0\}\}. \quad (10)$$

If all of τ_x, τ_y and τ_τ are not positive, it means that the switching of l_k does not occur. In such case, the trajectory must diverge. However, the situation never happens on the parameter conditions (6).

Step 3.

We calculate \mathbf{x}_{k+1} by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{a}(l_k) \cdot \tau_k. \quad (11)$$

We get the integer variable l_{k+1} after switching.

Step 4.

Let \mathbf{x}_{k+1} and l_{k+1} be replaced with \mathbf{x}_k and l_k , respectively. Subsequently, return to Step 2.

Here, we show an example of local maps in the case where l_k change from 0.

Table 1: Local vector fields and threshold for l

l	$B(T', \tau)$	$h(x)$	$\text{sgn}(y)$	$\mathbf{a}(l)$	$D_x(l)$	$D_y(l)$	$D_\tau(l)$
0	1	1	1	${}^t \begin{pmatrix} -\alpha - 1 & 1 + \beta & 1 \end{pmatrix}$	-1	0	$T'/2$
1	1	1	-1	${}^t \begin{pmatrix} -\alpha + 1 & 1 + \beta & 1 \end{pmatrix}$	-1	0	$T'/2$
2	1	-1	1	${}^t \begin{pmatrix} \alpha - 1 & -1 + \beta & 1 \end{pmatrix}$	1	0	$T'/2$
3	1	-1	-1	${}^t \begin{pmatrix} \alpha + 1 & -1 + \beta & 1 \end{pmatrix}$	1	0	$T'/2$
4	-1	1	1	${}^t \begin{pmatrix} -\alpha - 1 & 1 - \beta & 1 \end{pmatrix}$	-1	0	T'
5	-1	1	-1	${}^t \begin{pmatrix} -\alpha + 1 & 1 - \beta & 1 \end{pmatrix}$	-1	0	T'
6	-1	-1	1	${}^t \begin{pmatrix} \alpha - 1 & -1 - \beta & 1 \end{pmatrix}$	1	0	T'
7	-1	-1	-1	${}^t \begin{pmatrix} \alpha + 1 & -1 - \beta & 1 \end{pmatrix}$	1	0	T'

$$\mathbf{a}(0) = \begin{pmatrix} a_x(0) \\ a_y(0) \\ 1 \end{pmatrix} = \begin{pmatrix} -\alpha - 1 \\ 1 + \beta \\ 1 \end{pmatrix}. \quad (12)$$

- Switched $h(x)$

$${}^t \mathbf{n} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad D = D_x(0) = -1, \quad (13)$$

$$\mathbf{x}_{i+1} = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1+\beta}{-\alpha-1} & 1 & 0 \\ -\frac{1}{-\alpha-1} & 0 & 1 \end{pmatrix} \mathbf{x}_i + \begin{pmatrix} -1 \\ -1+\beta \\ -1-\alpha \end{pmatrix}. \quad (14)$$

- Switched $\text{sgn}(y)$

$${}^t \mathbf{n} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad D = D_y(0) = 0, \quad (15)$$

$$\mathbf{x}_{i+1} = \begin{pmatrix} 1 & -\frac{-\alpha-1}{1+\beta} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-1}{1+\beta} & 1 \end{pmatrix} \mathbf{x}_i. \quad (16)$$

- Switched $B(T', \tau)$

$${}^t \mathbf{n} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad D = D_\tau(0) = \frac{T'}{2}, \quad (17)$$

$$\mathbf{x}_{i+1} = \begin{pmatrix} 1 & 0 & -(-\alpha-1) \\ 0 & 1 & -(1+\beta) \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x}_i + \begin{pmatrix} (-\alpha-1)\frac{T'}{2} \\ (1+\beta)\frac{T'}{2} \\ \frac{T'}{2} \end{pmatrix}. \quad (18)$$

To analyze periodic orbit, we define Poincare map F_p .

$$S_p = \{\mathbf{x} | \tau = nT'\}, \quad (19)$$

$$F_p : S_p \rightarrow S_p, \quad (20)$$

$$(x_{n+1}, y_{n+1}, (n+1)T') = F_p(x_n, y_n, nT'). \quad (21)$$

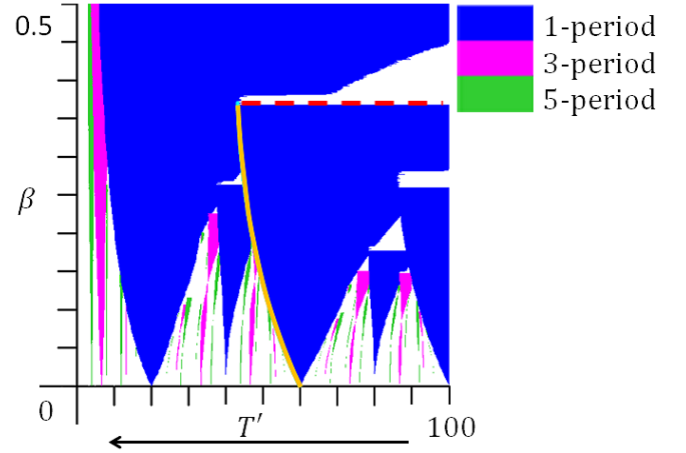


Figure 5: Bifurcation diagram($\alpha = 0.2$).

F_p is defined as composite mapping of local maps. For Poincare map F_p , we define period m .

$$(x_n, y_n, (n+m)T') = F_p^m(x_n, y_n, nT') \quad (22)$$

Bifurcation diagram is shown in Fig.5. Black arrow means flow of parameter change.

3.1. Bifurcation phenomena

We consider bifurcation phenomena when parameter values of β and T' are changed, respectively.

- Change β

Stability are determined by eigenvalues λ of Jacobian from Poincare map F_p . If moduli of λ are all less than 1, the periodic orbit is stable; otherwise, it is unstable. Eigenvalues of Fig.4(b) ($\beta = 0.2, T' = 60$) are

$$\lambda = \begin{pmatrix} 0.0308 - 0.5434i \\ 0.0308 + 0.5434i \\ 0 \end{pmatrix}. \quad (23)$$

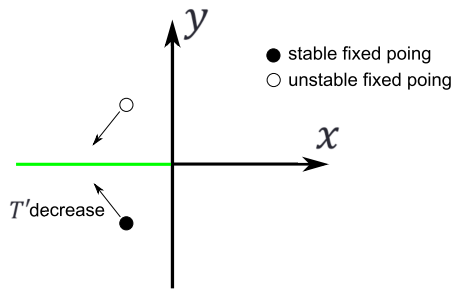


Figure 6: Movement of fixed points.

Eigenvalues of Fig.4(c) ($\beta = 0.37, T' = 50$) are

$$\lambda = \begin{pmatrix} 1.0366 \\ 0.0846 \\ 0 \end{pmatrix}. \quad (24)$$

Therefore, Fig.4(b) is stable and 4(c) is unstable. There is saddle-node bifurcation because one of eigenvalues crosses real value 1.

- Change T'

The fixed points are determined by

$$(x_n, y_n, (n+1)T') = F_p(x_n, y_n, nT'). \quad (25)$$

Movement of fixed points, when T' is decreased, is shown in Fig.6. Fixed point hits border $y = 0$ that switches $\text{sgn}(y)$. Then, border-collision bifurcation happens. From $y_n = 0$, bifurcation set of border-collision bifurcation is

$$T' = \frac{4(3\alpha^2 + 1)(\beta^2 - 1) - 32\alpha\beta}{(\beta^2 - 1)(\beta(3\alpha^2 + 1) + \alpha(\alpha^2 + 3))}. \quad (26)$$

On $y_n = 0$, Non-smooth saddle-node bifurcation happens because stable fixed point encounters unstable fixed point. Therefore, co-dimensional 2 bifurcation arises on the bifurcation set.

4. Conclusion

In the paper, we considered bifurcation phenomena that occur in a 2-D piecewise-constant oscillator. We proposed a novel method for piecewise-constant system with external force. By using the method, bifurcation phenomena from 2-D piecewise-constant driven oscillator were analyzed. Boundaries of Arnold tongues from the system were derived.

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