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An Energy Based Investigation of Rössler Type Chaos on Chua's Circuit

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Abstract—In order to deal with nonlinear systems, energy functions have been considered in many applications. In this paper, the aim is to set an approach based on energy functions to understand the mechanism behind the chaotic behavior of Chua's Circuit. In order to explain the effect of nonlinear resistor, different nonlinearities are considered and the simulation results based on energy functions are given.

1. Introduction

Euler-Lagrangian and Hamiltonian formalisms of circuits, especially nonlinear circuits, have been subject of interest for decades [1–3]. One reason for this interest is their versatility in explaining the behavior of mechanical systems and providing a sound approach to the stability analysis due to the consideration of energy function along with differential equations defining the dynamics of the system. As no general method can be developed for the analysis of nonlinear circuits like the well defined, easily applicable methods of linear circuit analysis, energy based approaches would give insight to understanding the behavior of nonlinear circuits.

The energy storage elements, namely capacitors and inductors characterize the dynamics of an electric circuit. So understanding energy exchange between these energy storage elements and other elements in an electric circuit would be functional in explaining the complex behavior of nonlinear circuits. This is the main motivation of energy-based techniques in nonlinear circuit theory [1]. Even though Lagrangian and Hamiltonian formulations are being used effectively in classical mechanics, this is not so easy in electric circuits as in circuit theory it is not clear how to define *potential energy* and *kinetic energy*. There are some studies in the literature to overcome this difficulty [2, 3].

Also energy point of view is effective in studying the control [4] and synchronization [5, 6] of the chaotic systems. In this work we will use energy function to investigate chaos producing mechanism in Chua's circuit. Chua's circuit is preferred for its simplicity and rich dynamic behavior and these attributes of the circuit made it very popular in chaos theory [7, 8].

Our approach will be to follow the exchange in the Hamiltonian of the nonlinear circuit along the trajectories and investigate energy exchange in the Chua's circuit. Nonlinear resistance N_R, is the only nonlinear element in Chua's circuit. If we replace it with a passive resistance there is no interesting dynamic in the circuit as the solutions decay in time and approaches zero [9]. To give an idea of how to use Hamiltonian function, we will take this case as our starting point and will examine the effect of the nonlinear resistor on the energy exchange. In this work we will examine the Rössler type chaos of Chua circuit from energy point of view.

This paper contains the following sections: In the following section Hamiltonian equations and Hamiltonian of the Chua's Circuit is derived. In Section III, the equations related to energy exchange of the circuit are given. Simulation results and related discussions for different Chua diodes are presented in Section IV. Finally, the conclusions are given in Section V.

2. Hamiltonian Equations of Chua's Circuit

The equations related to Chua's circuit, shown in Fig.1 are commonly given with the following state equations:

$$\begin{aligned} C_1\dot{v}_1 &= G(v_2 - v_1) - g(v_1) \\ C_2\dot{v}_2 &= i_3 - G(v_2 - v_1) \\ L\dot{i}_3 &= -v_2 \end{aligned} \qquad (1)$$

where

$$i_R = g(v_R) = G_b v_R + \frac{1}{2}(G_a - G_b)[|v_R + E| - |v_R - E|] \qquad (2)$$

is the node equation of the nonlinear resistor N_R.

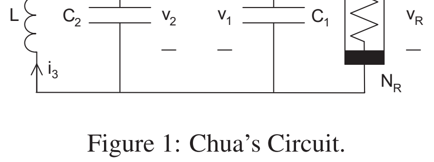


Figure 1: Chua's Circuit.

To derive Lagrangian of the circuit the method proposed by Chua and McPhearson [2] has been used. Following this method generalized coordinates (x) and generalized velocities ($\dot{x}$) are defined as follows:

$$x \overset{\Delta}{=} [\phi_1 \quad \phi_2]^T \quad (3a) \qquad \dot{x} \overset{\Delta}{=} [v_1 \quad v_2]^T \quad (3b)$$

where ϕ_1 and ϕ_2 are fluxes of the capacitors. We will omit details in deriving Lagrangian and Euler-Lagrange equations for simplicity. As a result Lagrangian becomes,

$$\mathcal{L} = \frac{1}{2}C_1 v_1^2 + \frac{1}{2}C_2 v_2^2 - \frac{1}{2L}\left[-\phi_2 + \phi_2(0) + \phi_3(0)\right]^2 \quad (4)$$

where

$$\phi_3(0) = Li_3(0). \quad (5)$$

One could derive Hamiltonian by using below equations:

$$\mathcal{H}(x, y) = y^T \dot{x} - \mathcal{L}(x, \dot{x}) \quad (6)$$

$$y \triangleq \frac{\partial \mathcal{L}}{\partial \dot{x}} \quad (7)$$

Here y denotes the generalized moments as given in [10]. Using equations (3b), (4), (6) and (7) $\mathcal{H}$ is derived as follows:

$$\mathcal{H} = \frac{1}{2}C_1 v_1^2 + \frac{1}{2}C_2 v_2^2 + \frac{1}{2L}\left[-\phi_2 + \phi_2(0) + \phi_3(0)\right]^2. \quad (8)$$

This is Hamiltonian of the Chua's circuit. In order to see that it is equal to sum of energies of the dynamic elements, it could be rewritten by using $L{-}C_2$ loop equation in Chua's circuit (Fig.1). With this manipulation we could get below equation which is in conventional form:

$$\mathcal{H} = \frac{1}{2}C_1 v_1^2 + \frac{1}{2}C_2 v_2^2 + \frac{1}{2}Li_3^2. \quad (9)$$

From this point to the end of the paper we will work with dimensionless Chua equations for simplicity. Dimensionless form of the Chua's equations are given as follows:

$$\begin{aligned} dx/d\tau &= \alpha(y - x - f(x)) \\ dy/d\tau &= x - y + z \\ dz/d\tau &= -\beta y \end{aligned} \quad (10)$$

where

$$f(x) = bx + \frac{1}{2}(a - b)[|x + 1| - |x - 1|] \quad (11)$$

is a piecewise linear function with 3-segments and is associated with Chua's diode. In order to obtain these equations, rescaling relations given in [7] are used. Actually using quantities m_0 and m_1 instead of a and b with the relations $m_0 = 1 + a$ and $m_1 = 1 + b$ gives slightly simpler equations [8]. However for our purpose using a and b is beneficial.

Hamiltonian can be rewritten as

$$\mathcal{H} = \frac{1}{2}\left[\frac{1}{\alpha}x^2 + y^2 + \frac{1}{\beta}z^2\right] \quad (12)$$

with a scale difference from (9) as given in [5].

3. Energy Surfaces for Double Scroll

Fig.2 gives an idea about how Hamiltonian is changing in the phase space. In the shaded regions where Hamiltonian is increasing; Chua's diode is providing energy to the circuit.

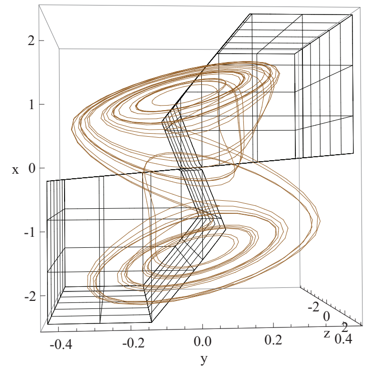


Figure 2: The regions Hamiltonian is increasing are denoted by shaded regions where paramater values are as (α, β, a, b)=(9, 100/7, −8/7, −5/7).

To determine the separating planes in Fig.2, one need to solve the equation $d\mathcal{H}/d\tau = 0$. By taking derivative of (12) and then substituting (10) into the the result yields the following equation

$$\frac{d\mathcal{H}}{d\tau} = -(x - y)^2 - xf(x). \quad (13)$$

where the term $f(x)$ in (13) has been given in (11).

In order to investigate the relation between the Chua's diode's characteristic and the energy exchange between the elements in the circuit, the characteristic of the diode will be considered further and each piecewise linear part will be considered step by step. In the upper side of Fig.3, (a) and (b), these cases are given. The equations for (a) and (b) are as follows, respectively:

$$f(x) = ax \quad (14a)$$

$$f(x) = \begin{cases} ax, & \text{for } x \geqslant -1 \\ bx - a + b, & \text{for } x < -1 \end{cases} \quad (14b)$$

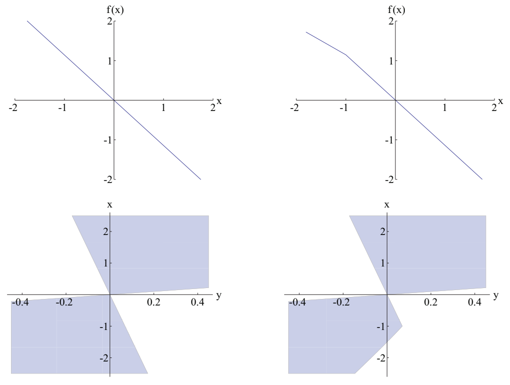
By using (13) and (14), one could derive the solution of the equation $d\mathcal{H}/d\tau = 0$. Since (14b) covers (14a) as special case we will give the result only for the case, (14b).

$$y = \begin{cases} x\left(1 \pm \sqrt{-a}\right), & \text{for } x \geqslant -1 \\ x \pm \sqrt{(a - b)x - bx^2}, & \text{for } x < -1. \end{cases} \quad (15)$$

These equations are separatrices for the regions where Hamiltonian is increasing. They are given in the lower side of Fig.3. In the next section each case will be investigated in detail.

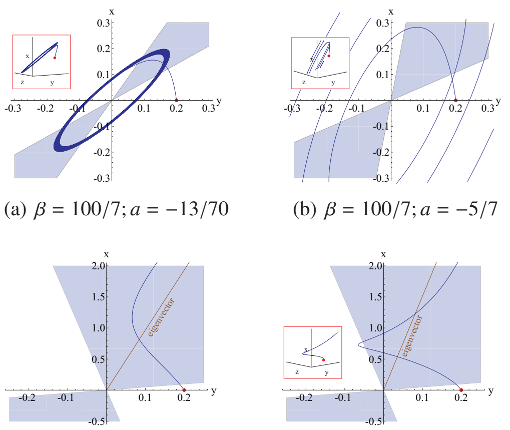
4. The Effect of Nonlinear Resistor on Energy Function

Since (13) and as a result of this, (15) are independent of z, $x - y$ plane of generalized velocities is sufficient to follow how Hamiltonian is changing. We will consider those seperatix planes and their effect on energy exchange.



(a) (b)

Figure 3: In the upper figure each column shows the characteristic of N_R and the lower figure shows related x-y projection of the regions where Hamiltonian is increasing.



(a) $\beta = 100/7; a = -13/70$ (b) $\beta = 100/7; a = -5/7$

(c) $\beta = 100/7; a = -8/7$ (d) $\beta = 23; a = -8/7$

Figure 4: x-y projections of phase plane for different values of bifurcation parameters β and a. Inset of (a), (b) and (d) shows 3D phase space.

4.1. N_R as Active Resistance

When the characteristic of N_R given in Fig.3(a) is used in Chua's circuit the only dynamic behavior that could be observed is equilibrium point. Trajectory goes to either zero or infinity. These cases are shown in Fig.4. The value of bifurcation parameter α is equal to 9 and initial conditions are as (x_0, y_0, z_0)=(0, 0.2, 0).

From this point to the end of the paper Chua's circuit will be simulated under below conditions for different parameter sets:

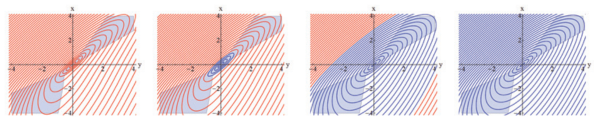
- Temporary dynamics will be excluded in the figures.
- Value of the system parameter α is fixed and it is 9.
- The values for the bifurcation paramaters a and b will be chosen such that the system has one real and a pair of complex conjugate roots for both parameters.
- Initial points will be chosen such that solutions won't be affected by unstable eigenvector. This dynamic behavior can be followed from Fig.4(c) and (d).

4.2. N_R as 2-segment Active Resistance

In the following different parameter values will be considered.

4.2.1. a=-13/70, b=-5/7

In Fig.5 the bifurcation dynamic of the circuit is given. For small values of β, system always gains energy, every trajectory diverges (see Fig.5(a)). While β increasing an unstable limit cycle occurs. System loses energy in inner region and gains energy in outer region. Energy exchange of the system is given for two different values of β in Fig.5 (b) and (c). Further, increasing the value of β causes limit cycle to extinct. In this case system always loses energy, every trajectory converges to an equilibrium point (see Fig.5(d)).



(a) $\beta \leqslant 13$ (b) $\beta = 14$ (c) $\beta = 23$ (d) $\beta \geqslant 24$

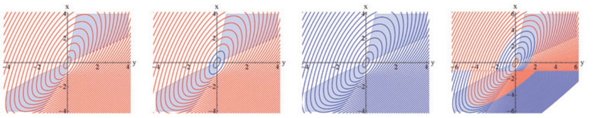
Figure 5: Energy exchange of the system. As the system parameter β grows, energy of the system is calculated each time the trajectory in the first quarter crosses the line $y = x\left(1 \pm \sqrt{-a}\right)$. When system gains energy red line is used and when system loses energy blue line is used.

4.2.2. a=-5/7, b=-13/70

In the previous case, we have observed that when inner region loses energy and outer region gains enery an unstable limit case occurs. Using this result, first insight may be swapping the values of the system parameters would be enough to create a stable limit cycle. Interestingly this is not enough. In Fig.6(a-c) the bifurcation dynamic of the circuit is given. To understand why system couldn't produce stable limit cycle, it is needed to examine energy exchange. In Fig.6(d) energy exchange of the system is plotted, each time the trajectory crosses the line $x = -1$. It is shown that, below the line system loses energy in the outer region. Hovewer system gains more energy than it loses in the above part of the outer region. If the system had lost more than it have gained, the aim would have been reached. In the next case we will change the system parameter a to obtain this.

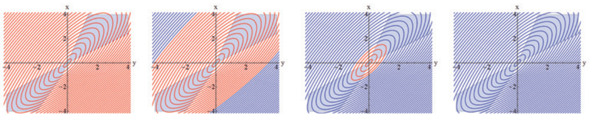
4.2.3. a=-3/7, b=-13/70

In Fig.7 the bifurcation dynamic of the circuit is given. As a comparision with previous case for $\beta = 21$ system gains less energy than it loses in the outer region and a stable limit cycle formed.



(a) $\beta \leqslant 23$ (b) $\beta = 23$ (c) $\beta \geqslant 23$ (d) $\beta = 21$

Figure 6: For different β values, energy exchange of the system.

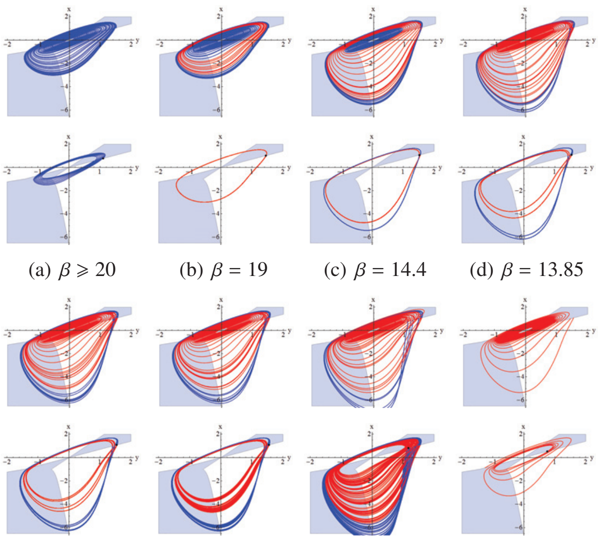


(a) $\beta \leqslant 20$ (b) $\beta = 21$ (c) $\beta = 23$ (d) $\beta \geqslant 24$

Figure 7: As the system parameter β grows, energy exchange of the system. In the figures color codes used are same with Fig.5.

4.2.4. a=-13/70, b=-8/7

In Fig.8 the bifurcation dynamic of the circuit is given. Starting with large β values, the system energy is always decreasing and as β values are decreased, three regions occur, where in the in-between region the energy increases. This case $\beta = 19$ corresponds to occurance of stable limit cycle. As β is decreased more, interference occurs between regions where energy is decreasing



(a) $\beta \geqslant 20$ (b) $\beta = 19$ (c) $\beta = 14.4$ (d) $\beta = 13.85$

(e) $\beta = 13.78$ (f) $\beta = 13.761$ (g) $\beta = 13.2$ (h) $\beta \leqslant 12$

Figure 8: As the system parameter β grows, dynamic of the system. For each case the upper figure shows the energy exchange while the lower shows solutions. In the figures color codes used are same with Fig.5.

and increasing in the outer regions. This phenomena keeps on as β is decreased more, interference occurs more; first period doubling then, with period doubling cascade or symmetry breaking chaotic behavior raises.

In the inner region, the energy could be either increasing or decreasing for different bifurcation parameter values, but this does not have an effect on dynamical behavior. If in the inner region energy is increasing while chaotic behavior occures this chaotic behavior is robust.

5. Conclusion

In this paper, the period doubling mechanism in Chua's circuit is investigated utilizing energy relations for Rössler type Chaos. Stepwise, first Hamiltonian of the circuit is obtained and then with different Chua diodes, the relation between the energy function and the dynamic behavior is observed. The expectation is to set a theoretical method to investigate chaos based on energy function.

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