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Recovery of Parameters and Connections in Networks of Time-Delay Systems

Mikhail Prokhorov†, Vladimir Ponomarenko†‡ and Ilya Sysoev‡

†Saratov Branch of the Institute of Radio Engineering and Electronics of Russian Academy of Sciences
Zelyonaya Street 38, Saratov 410019, Russia
‡Department of Nano- and Biomedical Technologies, Saratov State University
Astrakhanskaya Street 83, Saratov, 410012, Russia
Email: mdprokhorov@yandex.ru, ponomarenkovi@gmail.com, ivssci@gmail.com

Abstract– We propose a method that allows one to recover the parameters of elements and architecture of couplings in networks of time-delay systems from time series. The method is based on the reconstruction of model delay-differential equations for the network elements and diagnostics of statistical significance of couplings. It can be applied to networks composed of nonidentical units with an arbitrary number of unidirectional and bidirectional couplings. We verify our method using both numerical and experimental data.

1. Introduction

The ensembles of coupled delay-differential equations are widely used in recent years for modeling and description of processes in various networks with a time-delayed feedback. Studying these networks it is important to reconstruct both the parameters of units and architecture of connections from experimental time series. It is a complex problem, since even simple single time-delay systems can possess high-dimensional chaotic dynamics. For a successful recovery of time-delay systems one has to use special methods [1–3]. However, the most of them are intended for the reconstruction of model equations for single time-delay systems from time series. The presence of links between the time-delay units complicates the problem of reconstruction and calls for the development of new methods.

The problem of simultaneous estimation of the network connectivity and node parameters including the delay time for networks of time-delay systems has been addressed recently in [4]. However, this problem was solved in the absence of noise under the assumption that the node functions are invertible and the initial conditions for the unknown delays are chosen in a neighbor set of the true values. In this paper we propose a method for reconstructing the parameters of elements and architecture and strengths of couplings in networks of time-delay systems, which is devoid of the above mentioned limitations.

The paper is organized as follows. Section 2 contains the method description. In Sec. 3 the method is applied for the reconstruction of networks of time-delay systems from simulated and experimental time series. In Sec. 4 we summarize our results.

2. Method Description

Let us consider a network composed of coupled time-delay systems, each described by the equation

$$\varepsilon_i \dot{x}_i(t) = -x_i(t) + f_i(x_i(t-\tau_i)) + \sum_{j=1(j\neq i)}^{M} k_{i,j}(x_j(t) - x_i(t)), \quad (1)$$

where $x_i \in R$, $i = 1,\ldots,M$, M is the number of elements, τ_i is the delay time, the parameter ε_i characterizes the element inertial properties, $f_i : R \to R$ is a nonlinear function, and $k_{i,j}$ are the coupling coefficients characterizing the strength of influence $j \to i$, i.e., from the jth element to the ith one.

We propose an approach to the recovery of the element parameters and architecture of couplings in the network of time-delay systems, which involves two steps. At first we recover the delay time τ_i of each element. Then, knowing τ_i, we reconstruct the parameters ε_i and $k_{i,j}$ and nonlinear functions f_i.

2.1. Recovery of Delay Time of the Elements

In [2] we have shown that the time series of single ($k_{i,j} = 0$) time-delay systems (1) practically have no extrema separated in time by the delay time. If such systems perform chaotic oscillations, the extrema in their time series are located irregularly and the time intervals between these extrema can take different values. Taking into account this feature, a method for the delay time recovery has been proposed based on the statistical analysis of time intervals between extrema in the chaotic time series of time-delay system. Defining, for different values of τ, the number N_i of situations where the points of the time series separated in time by τ are both extremal, we can construct the $N_i(\tau)$ plot and recover the delay time τ_i as the value at which the absolute minimum of $N_i(\tau)$ is observed. The features of $N_i(\tau)$ plot and details of its calculation are explained in [2].

Let us consider how the presence of couplings between the time-delay systems influences on the efficiency of this method. The action of other elements in the network on the time-delay system under consideration disturbs it and results in the disappearance of some extrema in the system time series and appearance of new ones. Let us differentiate Eq. (1) with respect to t:

$$\varepsilon_i \ddot{x}_i(t) = -\dot{x}_i(t) + \frac{df_i\left(x_i(t-\tau_i)\right)}{dx_i(t-\tau_i)}\dot{x}_i(t-\tau_i) + \sum_{j=1(j\neq i)}^{M} k_{i,j}\left(\dot{x}_j(t) - \dot{x}_i(t)\right). \quad (2)$$

In the presence of inertial properties ($\varepsilon_i > 0$), which corresponds to real situations, the extrema in $x_i(t)$ are close to quadratic ones and therefore $\dot{x}_i(t) = 0$ and $\ddot{x}_i(t) \neq 0$ at the extremal points. If for $\dot{x}_i(t) = 0$ in a typical case $\ddot{x}_i(t) \neq 0$, then, for $\varepsilon_i \neq 0$ the condition

$$\frac{df_i\left(x_i(t-\tau_i)\right)}{dx_i(t-\tau_i)}\dot{x}_i(t-\tau_i) + \sum_{j=1(j\neq i)}^{M} k_{i,j}\dot{x}_j(t) \neq 0 \quad (3)$$

must be fulfilled. The second term in (3) is equal to zero in the case of absence of couplings ($k_{i,j} = 0$) and in the case of strong couplings ensuring the synchronization of elements, which results in $\dot{x}_j(t) = \dot{x}_i(t)$. But we have set $\dot{x}_i(t) = 0$ to derive the condition (3). Hence, in these boundary cases the first term in (3) is not equal to zero. By this is meant that the derivatives $\dot{x}_i(t)$ and $\dot{x}_i(t-\tau_i)$ do not vanish simultaneously, i.e., there must be no extremum in $x_i(t)$ separated in time by τ_i from a quadratic extremum. In the intermediate cases of weak and moderate couplings it is possible to find extrema in $x_i(t)$ separated in time by τ_i. However, numerical simulation has shown that generally the probability of such situation is less than the probability to find a pair of extrema separated in time by $\tau \neq \tau_i$. As the result, the $N_i(\tau)$ plot will have a minimum at $\tau = \tau_i$. Therefore, the qualitative features of the $N_i(\tau)$ plot are retained for system (1) in a wide range of coupling coefficient values.

2.2. Recovery of Other Parameters and Connections in the Network

After reconstruction of τ_i we recover the parameter ε_i, function f_i, and coupling coefficients $k_{i,j}$ of the ith time-delay system (1), having at the disposal the time series of oscillations of all elements in the network. To do this, we propose the following approach. Let us write Eq. (1) as

$$\varepsilon_i \dot{x}_i(t) + x_i(t) - \sum_{j=1(j\neq i)}^{M} k_{i,j}(x_j(t) - x_i(t)) = f_i(x_i(t-\tau_i)). \quad (4)$$

If one plots the dependence of the left-hand side of Eq. (4) on $x_i(t-\tau_i)$, it will reproduce the function f_i. Since the parameters ε_i and $k_{i,j}$ are *a priori* unknown, we will search for them by minimizing the function

$$L_i(\varepsilon_i, k_{i,j}) = \sum_{n=1}^{S-1}\left(\left(y_{i,n+1} - y_{i,n}\right)^2 + \left(z_{i,n+1} - z_{i,n}\right)^2\right), \quad (5)$$

which characterizes the distance between the points in the (y_i, z_i) plane ordered with respect to y_i. Here $y_i = x_i(t-\tau_i)$, $z_i = \varepsilon_i \dot{x}_i(t) + x_i(t) - \sum_{j=1(j\neq i)}^{M} k_{i,j}\left(x_j(t) - x_i(t)\right)$, n is the point serial number, and S is the number of points. In the case of incorrect choice of ε_i and $k_{i,j}$, the points in the (y_i, z_i) plane do not lie on a single-valued curve f_i. Hence, the $L_i(\varepsilon_i, k_{i,j})$ value is greater than that for true ε_i and $k_{i,j}$.

We set the initial conditions for ε_i and $k_{i,j}$ and then refine them by the Nelder-Mead method minimizing the function (5), which minimum is denoted as $L_{i,M}$. At $M > 4$ the situation in which the method fails to reveal the nonexisting couplings becomes typical. These couplings are detected as weak ones because of indirect couplings via other elements. To reject insignificant couplings we use the method of successive trial elimination of coefficients $k_{i,j}$ from the model (1). We advance the hypothesis that the coupling $j \to i$ is absent, eliminate the corresponding coupling coefficient $k_{i,j}$, and reconstruct the other parameters of the model minimizing the function (5), which minimum is denoted as $L_{i,j,M-1}$. This procedure is then repeated by eliminating another $k_{i,j}$ at the fixed i, and so on for all $j \neq i$. Note that at each step we assume that the ith element is not affected by only one of jth elements. Finally, we determine the elimination of which $k_{i,j}$ from (1) yields $L_{i,M-1} = \min_j L_{i,j,M-1}$ and estimate the statistical significance of $L = L_{i,M-1}/L_{i,M}$. In doing this we are guided by the following arguments. At large S, the differences $y_{i,n+1} - y_{i,n}$ and $z_{i,n+1} - z_{i,n}$ in (5) are distributed according to the distribution that is close to the normal one. Here $S/2$ of these differences can be considered as independent because they have no common coordinates. Besides, $L_{i,M}$ depends on M parameters of model (4). This fact reduces the number of independent quantities in (5) to $S/2 - M$. Then, taking into account that a sum of the squares of K independent normal random variables is distributed according to the chi-square distribution with K degrees of freedom [5], we obtain that the $L_{i,M}$ values calculated at different parameters and/or in the presence of noise are distributed according to the chi-square distribution with $S/2 - M$ degrees of freedom and the $L_{i,M-1}$ values are distributed according to the chi-square distribution with $S/2 - M + 1$ degrees of freedom.

If X is a ratio of two independent random variables distributed according to the chi-square distribution with v and w degrees of freedom, respectively, then it has the Fisher–Snedecor distribution with the distribution function

$$F_{v,w}(X) = B_d\left(\frac{v}{2}, \frac{w}{2}\right), \quad (6)$$

where B is the regularized incomplete beta function and $d = vX/(vX + w)$ [6]. Hence, L has the distribution function (6) with $X = L$, $v = S/2 - M + 1$, and $w = S/2 - M$.

We denote the value of L for which $F_{v,w}\left(L_{1-p}\right)=1-p$, where p is the statistical significance level, as L_{1-p}. Then, if $L > L_{1-p}$, one can conclude at a significance level p that the ith element is affected by all other elements of the ensemble, i.e., all $k_{i,j} \neq 0$. In the opposite case, we conclude that the coupling $j \to i$ is absent and check the significance of other couplings, successively eliminating one by one the remaining links from other elements to the ith one. The procedure is repeated until all couplings become significant. This approach allows one to recover the coupling architecture, parameters of all elements, and nonlinear functions.

If the number of connections between the network elements is known to be small, it is preferably to use the method of successive trial addition of coefficients $k_{i,j}$ to the model for the reconstruction of architecture and strengths of couplings. First we find the minimum $L_{i,1}$ of function (5) assuming that all $k_{i,j}$ are absent in Eq. (1), i.e., there are no couplings. Then we enter one coefficient $k_{i,j}$ into (1) and find the minimum of function (5), which is denoted as $L_{i,j,2}$. This procedure is then repeated by entering another $k_{i,j}$ at the fixed i, and so on for all $j \neq i$. Finally, we determine the entering of which $k_{i,j}$ into the model yields $L_{i,2} = \min_j L_{i,j,2}$. If $L > L_{1-p}$, where $L = L_{i,1}/L_{i,2}$ has the distribution function (6) with $v = S/2-1$ and $w = S/2-2$, then the introduced coupling is nonzero at a significance level p. The procedure is repeated until the next coupling entered into the model turns out to be insignificant.

3. Method Application

3.1. Recovery of Network of Ikeda Equations

Let us reconstruct the parameters of elements and coupling architecture in a network of Ikeda equations:

$$\dot{x}_i(t) = -x_i(t) + \mu_i \sin\left(x_i(t-\tau_i) - x_{0i}\right) + \sum_{j=1(j\neq i)}^{M} k_{i,j}\left(x_j(t) - x_i(t)\right). \quad (7)$$

Eq. (7) is a special case of Eq. (1) with $\varepsilon_i = 1$. Fig. 1(a) shows the coupling architecture generated randomly in a network of 10 elements. There are 40 couplings from 90 possible ones. The parameters of the elements are assigned the arbitrary values in the following ranges: $\tau_i \in [2,5]$, $\mu_i \in [15,25]$, $x_{0i} \in [0,2\pi]$, and $k_{i,j} \in [0.1,0.5]$. In this case all the elements exhibit chaotic oscillations. Besides, each element is affected by independent white Gaussian noise with a zero-mean and variance $\sigma_i^2 = 0.04$.

We illustrate the results of reconstruction of one of the elements with the parameters $\tau_7 = 2.15$, $\mu_7 = 21.67$, $x_{07} = 3.88$, $k_{7,4} = 0.445$, $k_{7,6} = 0.172$, $k_{7,9} = 0.311$, $k_{7,10} = 0.435$, and $k_{7,j} = 0$, $j = 1,2,3,5,8$. For various τ values we count the number N_7 of situations where $\dot{x}_7(t)$ and $\dot{x}_7(t-\tau)$ are simultaneously equal to zero and construct the $N_7(\tau)$ plot [Fig. 1(b)]. For the step of τ variation equal to 0.01, the minimum of $N_7(\tau)$ is observed at the true delay time $\tau = \tau_7 = 2.15$. To construct this plot we use 40000 points of the time series, which exhibits about 1600 extrema.

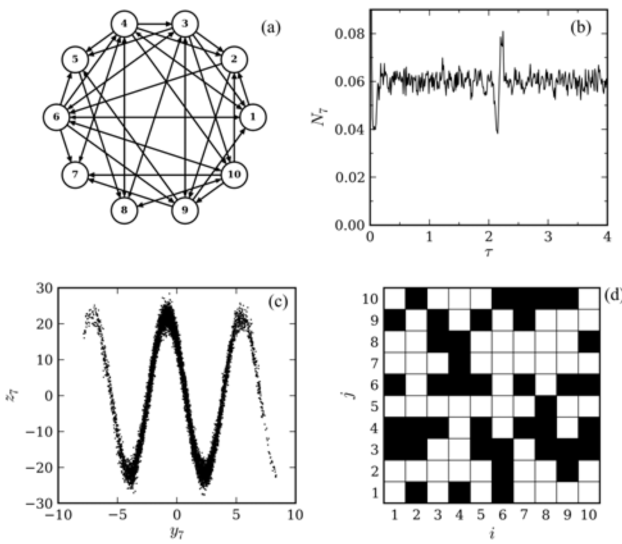

Fig. 1. (a) Scheme of connections in a network. (b) Dependence $N_7(\tau)$. $N_{7\min}(\tau) = N_7(2.15)$. (c) Function f_7 recovered in the plane (y_7, z_7), where $y_7 = x_7(t-\tau_7')$ and $z_7 = \dot{x}_7(t) + x_7(t) - \sum_{j=1(j\neq 7)}^{10} k'_{7,j}\left(x_j(t) - x_7(t)\right)$. (d) Results of estimation of the network connectivity.

Fig. 1(c) shows the nonlinear function f_7 recovered using the method of successive trial addition of coupling coefficients to the model at $p = 0.05$. The function f_7 is constructed at the recovered parameters $\tau_7' = 2.15$, $k'_{7,4} = 0.517$, $k'_{7,6} = 0.188$, $k'_{7,9} = 0.355$, $k'_{7,10} = 0.490$, and $k'_{7,j} = 0$, $j = 1,2,3,5,8$. The approximation of the recovered function f_7 with a harmonic function gives us the parameter estimation $\mu_7' = 22.00$ and $x_{07}' = 3.97$. Similarly the parameters and coupling coefficients of other elements are reconstructed. The results of recovery of coupling architecture are presented in Fig. 1(d). A square with a horizontal coordinate i and a vertical coordinate j shows the influence $j \to i$, except for the squares in the diagonal, which carry no information. All 40 existing couplings are detected at the significance level $p = 0.05$ (black squares) employing either the method of addition of couplings or the method of successive trial elimination of coupling coefficients from the model. The lengths of time series used for constructing Figs. 1(c) and (d) are 10000 points.

3.2. Recovery of coupled experimental electronic oscillators with time-delayed feedback

We apply the method to experimental time series gained from three coupled electronic oscillators with time-delayed feedback. Fig. 2(a) shows a block diagram of the experimental setup involving three coupled oscillators, each comprising a delay line, a nonlinear device, and a low-frequency first-order *RC* filter. The delay lines and nonlinear devices are implemented on microcontrollers, while the filters are analog devices. The digital and analog elements of this scheme are linked via the corresponding analog-to-digital converters (ADC) and digital-to-analog converters (DAC). The oscillators are coupled via resistors R_c.

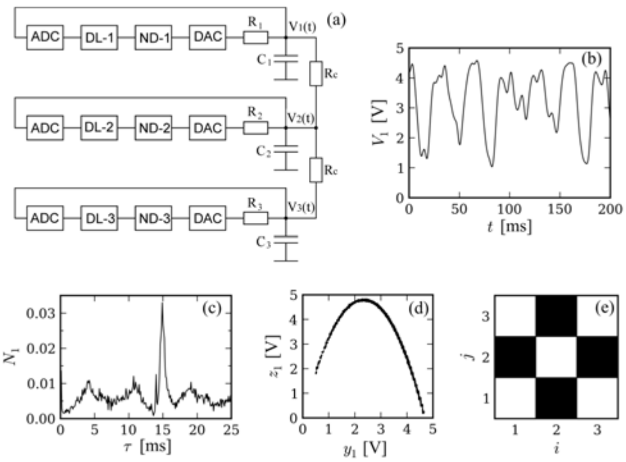

Fig. 2. (a) Block diagram of the experimental setup. DL are the delay lines and ND are the nonlinear devices, (b) The chaotic time series of $V_1(t)$. (c) Dependence $N_1(\tau)$. $N_{1\min}(\tau) = N_1(13.6\text{ ms})$. (d) Function f_1 reconstructed in the plane (y_1, z_1), where $y_1 = V_1(t-\tau_1')$ and $z_1 = \varepsilon_1'\dot{V}_1(t) + V_1(t) - \sum_{j=2}^{3} k'_{1,j}\left(V_j(t) - V_1(t)\right)$. (e) Diagram for coupling estimation results.

A model equation for the *i*th element in the network is as follows:

$$R_i C_i \dot{V}_i(t) = -V_i(t) + f_i\left(V_i(t-\tau_i)\right) + \sum_{j=1(j\neq i)}^{M} k_{i,j}\left(V_j(t) - V_i(t)\right), \tag{8}$$

where $V_i(t)$ and $V_i(t-\tau_i)$ are the delay line input and output voltages, respectively, τ_i is the delay time, R_i and C_i are the resistance and capacitance, respectively, and f_i is the transfer function of the nonlinear device. Eq. (8) is of form (1) with $\varepsilon_i = R_i C_i$.

All the nonlinear devices have a quadratic transfer function. We record the chaotic signals $V_i(t)$ of three nonidentical oscillators using a three-channel ADC with the sampling frequency equal to 10 kHz. Fig. 2(b) shows a part of the time series of $V_1(t)$ in the first oscillator having the parameters $\tau_1 = 13.6$ ms, $\varepsilon_1 = 2.88$ ms, $k_{1,2} = R_1/R_c = 0.1$, and $k_{1,3} = 0$.

For the step of τ variation equal to the sampling time $T_s = 0.1$ ms, the absolute minimum of $N_1(\tau)$ takes place at $\tau = 13.6$ ms [Fig. 2(c)]. The function f_1 recovered from experimental time series using the method of addition of coupling coefficients to the model at $p = 0.05$ is presented in Fig. 2(d). This function is constructed at the recovered parameters $\tau_1' = 13.6$ ms, $\varepsilon_1' = 2.74$ ms, $k_{1,2}' = 0.098$, and $k_{1,3}' = 0$. It coincides closely with the true transfer function f_1 of the nonlinear device of the first oscillator. The method of elimination of couplings gives the same results.

4. Conclusion

We have proposed the method for recovering the parameters of elements and architecture of connections in networks of time-delay systems from time series. The procedure of reconstruction involves two steps. At first we recover the delay time of each element using the method based on the statistical analysis of extrema in time series. At the second step we reconstruct the other parameters and architecture of couplings using the method based on the recovery of model equations for the network elements and diagnostics of statistical significance of couplings. To test the significance of links we employ the method of successive trial elimination or addition of couplings.

Acknowledgments

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References

[1] R. Hegger, M. J. Bünner, H. Kantz, A. Giaquinta, “Identifying and modeling delay feedback systems,” *Phys. Rev. Lett.*, vol.81, pp.558–561, 1998.

[2] M. D. Prokhorov, V. I. Ponomarenko, A. S. Karavaev, B. P. Bezruchko, “Reconstruction of time-delayed feedback systems from time series,” *Physica D*, vol.203, pp.209–223, 2005.

[3] L. Zunino, M. C. Soriano, I. Fischer et al., “Permutation-information-theory approach to unveil delay dynamics from time-series analysis,” *Phys. Rev. E*, vol.82, 046212, 2010.

[4] X. Wu, Z. Sun, F. Liang, C. Yu, “Online estimation of unknown delays and parameters in uncertain time delayed dynamical complex networks via adaptive observer,” *Nonlinear Dyn.*, vol.73, pp.1753–1768, 2013.

[5] M. Kendall, A. Stuart, The Advanced Theory of Statistics, New York: MacMillan, 1979.

[6] N. L. Johnson, S. Kotz, N. Balakrishnan, Continuous Univariate Distributions, vol. 2, New York: Wiley, 1995.