Looking for the network structural properties that facilitate fast social consensus

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Abstract—The social consensus formation process can be modeled by individuals exchanging opinions in a networked environment. This paper discusses the structural properties of social networks that affect the rate of consensus formation. Particularly, it is found that the degree distribution and degree mixing pattern of a social network play important roles in accelerating or decelerating the consensus formation process.

1. Introduction

A social network consists of a group of individuals who interact with each other. Social consensus is the agreement on certain opinion by members of the social network. Understanding how the structure of social networks affects the consensus formation process is essential in organization management and decision making process. Recent study models social networks as complex networks where individuals are connected through social relationships and can exchange opinions with each other under a set of rules.

In this paper, we study the consensus process on networks of different topological structures using DeGroot model [1]. It is shown that the consensus rate can be directly related to the second largest eigenvalue of the transformation matrix underlying the social network [2]. However, the second largest eigenvalue does not have a convenient physical meaning in the context of social network. Therefore, from a more practical aspect, we analyze how the degree distribution and degree mixing pattern of the network can affect the rate of social consensus formation process.

2. Social Consensus and DeGroot Model

Consider a social community with N members ($i = 1, ..., N$), the opinions of the members at time t can be written as $X(t) = \{x_1(t), x_2(t), ..., x_n(t)\} \in \mathbb{R}^n$. The members exchange opinions iteratively with each other and update their opinions at each time step. Finally, the opinions of all members in the community may converge. The strict definition of consensus is $\|x(t) - x^*\| = 0$, where x^* is the final consensus value which eventually every member agrees on and $\|...\|$ is the norm of a vector. In practice, the opinions of all members may take infinite time to converge to x^*. Hence, a more practical definition of consensus can be

$$\|x(t) - x^*\| \leq \epsilon, \tag{1}$$

where ϵ is either an arbitrarily small positive quantity ($\epsilon \to 0$) or a real number larger than 0 [2]. Define τ as the earliest time such that Eqn.(1) holds for all $t \geq \tau$. Then, τ is the time the system approaches practical consensus.

In this paper, we use DeGroot model to simulate the consensus formation process in social networks. In this model, individuals are connected by directed edges to form a complex network. At each time step t, an individual i update its opinion $x_i(t)$ by the weighted average opinion of its neighbors, including itself, i.e.,

$$x_i(t) = \sum_j w_{ji} x_j(t-1), \tag{2}$$

where j is any individual that is connected to i by a directed edge $j \to i$, w_{ji} is the weight of this edge and $\sum w_{ji} = 1$. The DeGroot model is also commonly used in control system to model the consensus process of robotic agents [3]. It is shown that in a directed network, the opinions of individuals converge if every set of nodes that is strongly connected and closed is aperiodic [1], or if the network has a spanning tree [4]. A social network may not fulfill the strict convergence condition, but in most large social networks, the opinions of individuals almost always approach practical consensus [5].

The DeGroot model is actually the process of applying a transform matrix T to the vector of individual's opinions X, i.e., $X(t) = T \cdot X(t-1)$. The transform matrix T is associated with the connection matrix A and the weights of edges from the underlying network. The eigenvectors of T are the non-zero vectors that, after being multiplied by T, remain parallel to the original vector. For each eigenvector, the corresponding eigenvalue is the factor by which the eigenvector is scaled when multiplied by the matrix. Hence if the system converges, the largest eigenvalue of the transform matrix T is 1. Then the real part of the second largest eigenvalue (of the eigenvector which scales slowest) of T decides the lower bound of the consensus speed [2, 6].

However, in a networked system, it is very difficult to find the physical meaning of the real part of the second largest eigenvalue of the underlying network. Therefore, in order to facilitate better organization management and decision making process, we need to find the network properties those can affect the consensus formation process in a practical way.

3. Impact of Degree Distribution to Consensus Rate

In the DeGroot model, an individual updates its opinion by the weighted average opinion of its neighbors. This process is equivalent to each individual influencing its neighbors in every iteration. The influence $f_i(t)$ of individual i to the network at time t can be written as

$$f_i(t) = \sum_j w_{ji} x_i(t), \tag{3}$$

where node j is any individual that is connected to i through an edge $i \rightarrow j$. Also, w_{ij} is the weight of the edge pointing from node j to node i. In the simplest situation, $w_{ij} = 1/k_i$ and $w_{ji} = 1/k_j$, where k_i and k_j are the degrees of nodes i and j.

In order to study the impact of degree distribution to the consensus rate, we first simplify a complex network to a smaller one where nodes are grouped by their degrees and each group of nodes interact with other groups as a whole. In a directed network, nodes can be grouped by their in-degrees k_{in} and out-degrees k_{out}. The average opinion $x_{k_{\text{in}}}$ of nodes of in-degree k_{in} is updated by:

$$x_{k_{\text{in}}}(t) = \frac{1}{N_{k_{\text{in}}}} \sum_{k_{\text{out}}} M_{k_{\text{out}}k_{\text{in}}} \frac{x_{k_{\text{out}}}(t-1)}{k_{\text{in}}}, \tag{4}$$

where $N_{k_{\text{in}}}$ is the number of nodes of in-degree k_{in}, $M_{k_{\text{out}}k_{\text{in}}}$ is the total number of edges directed from nodes of out-degree k_{out} to nodes of in-degree k_{in} and $x_{k_{\text{out}}}$ is the average opinion of nodes of out-degree k_{out}. The average influence of nodes of out-degree k_{out} can be written by:

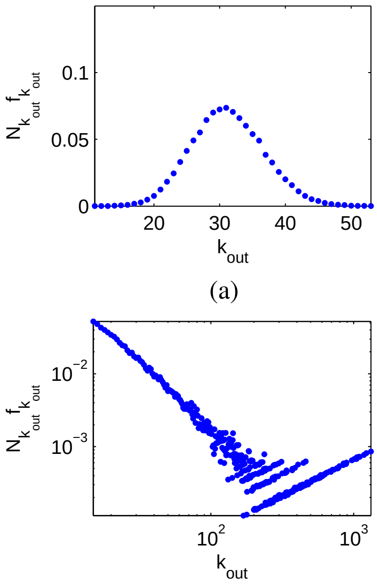
$$f_{k_{\text{out}}}(t) = \frac{1}{N_{k_{\text{out}}}} \sum_{k_{\text{in}}} M_{k_{\text{out}}k_{\text{in}}} \frac{x_{k_{\text{out}}}(t)}{k_{\text{in}}}, \tag{5}$$

where $N_{k_{\text{out}}}$ is the number of nodes of out-degree k_{out}. Let $P_e(k_{\text{out}})$ be the probability that any edge in the network is directed from a node of out-degree k_{out}, i.e.,

$$P_e(k_{\text{out}}) = \frac{M_{k_{\text{out}}}}{M}, \tag{6}$$

where $M_{k_{\text{out}}}$ is the total number of edges directed from nodes of out-degree k_{out} and M is the total number of edges in the network. Also, let $P_e(k_{\text{out}}|k_{\text{in}})$ be the probability that an edge pointing to a node of in-degree k_{in} is directed from a node of out-degree k_{out}, i.e.,

$$P_e(k_{\text{out}}|k_{\text{in}}) = \frac{M_{k_{\text{out}}k_{\text{in}}}}{M_{k_{\text{in}}}}, \tag{7}$$



(b)

Figure 1: The accumulated influence $N_{k_{\text{out}}} f_{k_{\text{out}}}$ of each group of individuals of out-degree k_{out} in (a) random network and (b) BA network.

where $M_{k_{\text{in}}}$ is the total number of edges pointing to nodes of in-degree k_{in}. If the mixing of nodes of different degrees in the network is random, then $P_e(k_{\text{out}}) = P_e(k_{\text{out}}|k_{\text{in}})$ for all k_{in} [7], i.e.,

$$\frac{M_{k_{\text{out}}}}{M} = \frac{M_{k_{\text{out}}k_{\text{in}}}}{M_{k_{\text{in}}}}. \tag{8}$$

Assuming the degree mixing is random in the network, the influence of any node of out-degree k_{out} can be written as

$$f_{k_{\text{out}}}(t) = \frac{k_{\text{out}}}{\langle k_{\text{out}} \rangle} x_{k_{\text{out}}}(t). \tag{9}$$

Therefore, the expected opinion update of nodes of in-degree k_{in} depends only on the out-degree distribution of the network, while the expected influence of a node is proportional only to its out-degree k_{out}. If we assume the opinions of individuals in each group the same, then the average opinion of all the individuals in the network is influenced most strongly by the degree group with the largest accumulated influence $N_{k_{\text{out}}} f_{k_{\text{out}}}$.

The impact of network degree distribution to the consensus rate is elaborated through simulation of DeGroot model in two networks with different degree distributions, i.e., a random network and a BA network generated using the preferential attachment model [8]. The accumulated influence of each group of nodes is shown in Fig. 1. It can be found that in the random network, nodes with medium numbers of connections contribute most of the influence to the network while in the BA network, nodes with the largest and the smallest number of connections contribute most of the influence to the network.

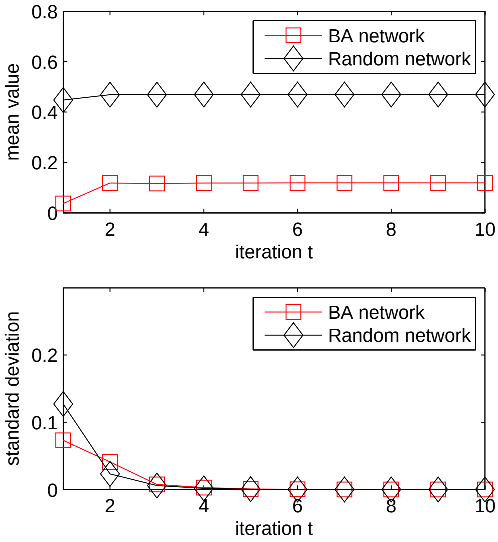


Figure 2: The first 10 steps of the consensus process on random network and BA network. (a) Convergence of mean value of all opinions. (b) Standard deviation of all opinions.

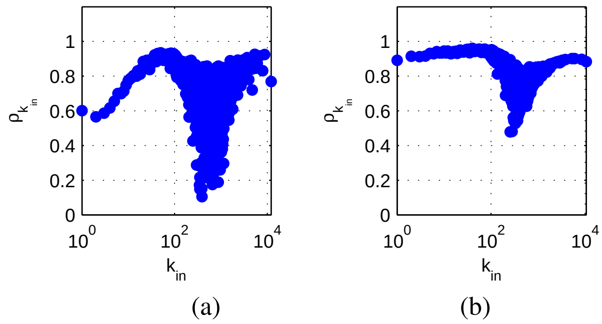


Figure 3: Degree mixing correlation $\rho_{k_{\text{in}}}$ between $P_e(k_{\text{out}})$ of the network and $P_e(k_{\text{out}}|k_{\text{in}})$ versus k_{in} in (a) the Twitter user network, (b) the randomized network.

The initial opinions of individuals $X(0)$ in different degree groups are set as

$$x_{k_{\text{out}}} = \frac{\text{rank of } k_{\text{out}}}{\text{maximum rank of } k_{\text{out}}}, \tag{10}$$

where the out-degrees in the network are sorted in an ascending order, e.g., a larger degree is ranked higher. The generated opinion is between 0 and 1. We compare the mean value and the standard deviation of every individual's opinion $X(t)$ as the DeGroot model iterates on the network. Fig. 2 shows that opinions on random network converge faster than those on BA networks. This is due to that the majority of individual's influence to the random network are concentrated from nodes with medium numbers of degrees while in the BA network, individuals with largest numbers of connections and those with smallest numbers of connections but of great quantity differ in their initial opinions.

The degree distribution of a social network affects the consensus rate in such a way that if several degree groups of individuals have strong influence to the network, the more similar their opinions are, the faster the consensus converges in the social network. Under our configuration of initial opinions in Eqn.(10), networks with long-tail degree distribution will always suffer from slower consensus rate than networks with random degree distribution.

4. Impact of Degree Mixing Pattern to Consensus Rate

In the previous section, we have defined the $P_e(k_{\text{out}}|k_{\text{in}})$ as the probability that an edge pointing to a node of in-degree k_{in} is directed from a node of out-degree k_{out}. $P_e(k_{\text{out}}|k_{\text{in}})$ also represents the proportion of information received from nodes of out-degree k_{out} to nodes of in-degree k_{in}. If $P_e(k_{\text{out}})$ is consistent with $P_e(k_{\text{out}}|k_{\text{in}})$, the nodes of in-degree k_{in} can be considered receiving information fairly from all the nodes in the network. If $P_e(k_{\text{out}})$ is not consistent with $P_e(k_{\text{out}}|k_{\text{in}})$, the nodes of in-degree k_{in} can be considered receiving biased information. A mixing correlation $\rho_{k_{\text{in}}}$ is defined as the Pearson's correlation coefficient between $P_e(k_{\text{out}}|k_{\text{in}})$ and $P_e(k_{\text{out}})$ for each group of nodes of in-degree k_{in} [5]. A $\rho_{k_{\text{in}}}$ being equal to 1 means that for nodes of in-degree k_{in}, the probability that any of their adjacent edge is directed from nodes with out-degree k_{out} is the same as the probability that any edge in the network is directed from a node of out-degree k_{out}. A $\rho_{k_{\text{in}}}$ being equal to 0 means that for nodes of in-degree k_{in}, the probability that any of their adjacent edge is directed from nodes with out-degree k_{out} has nothing to do with the degree distribution of the network.

In random network models, the degree mixing is by definition random [9]. In order to find a social network with non-random degree mixing pattern, we have obtained a user network from the online social service Twitter. The Twitter user network consists of more than 50,000 nodes and 1.5 million edges. Fig. 3(a) shows the degree mixing correlation $\rho_{k_{\text{in}}}$ in the Twitter user network. The non-random degree mixing pattern in the Twitter user network can be randomized through iterative edge switching process. First, two edges with distinct sources and destinations are selected. Then, the destinations of the two directed edges are switched. The process is repeated until the network is properly randomized. The randomized network has degree distributions identical to those of the original network. Fig. 3(b) shows the degree mixing correlation $\rho_{k_{\text{in}}}$ in the randomized Twitter user network. It can be found that the randomized Twitter user network has a more random degree mixing pattern than the original Twitter user network.

The DeGroot model of social consensus is simulated in both the Twitter user network and its randomized networks. The initial opinions of individuals are set using Eqn.(10).

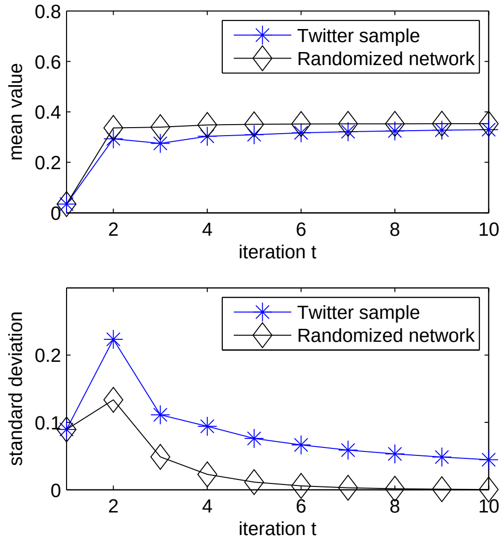


Figure 4: The first 10 steps of the consensus process on Twitter user network and its randomized network. (a) Convergence of mean value of all opinions. (b) Standard deviation of all opinions.

Fig. 4 shows that the consensus formation process is faster in the randomized network than in the Twitter user network. Compared to a social network with non-random degree mixing pattern, the influence that nodes in different degree groups receive are more likely to be identical in a network with random degree mixing pattern. Therefore, the non-random degree mixing pattern can hinder the consensus speed of the network.

5. Conclusion and Discussion

In this paper, we have introduced the social consensus problem and discussed the network properties that may affect the consensus convergence rate. The social consensus problem can be formulated by the DeGroot model. It is shown that the real part of the second largest eigenvalue of the transform matrix T of the network governs the consensus speed. Given a network topology that guarantees consensus, some carefully selected edge weight settings may let the system converge faster than other settings. Algorithms optimizing the transform matrix of the social network can find a certain configuration of edge weights that facilitate fastest consensus rate [10]. However, neither does the eigenvalue nor the optimized transform matrix have a convenient physical meaning in real life.

Therefore, we have analyzed the impact of network properties on the consensus rate in social networks from a practical perspective. Particularly, we have found that networks with degree distributions that results in a single group of nodes with strong influence reach consensus faster than networks with several groups of nodes with differed strong influence to the network. Furthermore, we have studied the mixing pattern between each group of nodes having the same degrees. Individuals in a network with random degree mixing pattern receive identical influence from others hence reach consensus faster than individuals in a network with non-random degree mixing pattern. Our findings may provide a set of practical guidelines to organization managers or politicians for manipulating organizational or social structure in order to promote efficient consensus formation process.

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