2010 International Symposium on Nonlinear Theory and its Applications
NOLTA2010, Krakow, Poland, September 5-8, 2010

Horseshoe Chaos in the Nano–Electro–Mechanical Casimir Nonlinear Oscillator

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Abstract—The progressive miniaturization of electronics components makes today feasible to exploit the nano–scale level. At this scale, phenomena peculiar of the quantum world arise, which can be exploited to realize innovative and potentially breakthrough devices. In this paper, the dynamic behavior of one such device, the Casimir nonlinear oscillator is analyzed. It is shown that, due to the nonlinear nature of the Casimir force, the oscillator can exhibit Smale horseshoes, and the mechanism leading to their birth is revealed.

1. Introduction

The progressive reduction in size of electronic devices, that in about 50 years has covered the interval from centimeters to microns, is now undergoing a further dramatic descent, from micro to nano–scale. At this scale, the laws of physics are quantum mechanical in nature, and new amazing phenomena, which are unexpected from a classical perspective, emerge.

An important prediction of quantum electrodynamics (QED) is the existence of irreducible fluctuations of the electromagnetic field even in vacuum. These fluctuations are responsible of van der Waals forces between atoms, and of Casimir forces, e.g. interactions between electrically neutral and highly conductive metals [1].

The boundary conditions imposed on the electromagnetic fields by the presence of metallic surfaces lead to a spatial redistribution of the mode density with respect to free space, creating a spatial gradient of the zero-point energy density and hence a net force between the metals [2]. Apart from its intrinsic relevance from the point of view of theoretical physics, the Casimir effect has recently attracted considerable attention for its possible engineer applications. Because boundary conditions can be tailored, this raises the interesting possibility of designing QED forces for specific applications, exploiting the fascinating idea to use the vacuum[1] as a device itself. In this optic, nano–electrometers, actuators, resonators and nonlinear oscillators have been realized and are under investigation [3, 4, 5, 6, 7].

The Casimir forces are inherently mesoscopic, since they can acquire significant values when the separation between the metallic surfaces is reduced to less than 100 nm, and nonlinear in nature. On the one hand, the mesoscopic nature allows a classical description of the dynamic behavior of the aforementioned devices. On the other hand, the nonlinear nature suggests the possible emergence of complex nonlinear behaviors. While there is vast experimental literature about hysteretic response and bistability of nonlinear oscillators in quantum optics, solid-state physics, mechanics, and electronics, it was only in [7] that the experimental observation of such phenomena caused by QED effects was given.

In this paper, we study the dynamical behavior of the Casimir nonlinear oscillator in the weakly damped, weakly forced regime. We show that, for some range of the parameters, the system has a homoclinic loop. By using the method of Melnikov, we prove that, under the effect of a periodic forcing, the system can exhibit transverse homoclinic orbits, and thus Smale horseshoes.

2. The Casimir nonlinear oscillator

A simple model of the Casimir oscillator is shown in figure 1. It is composed of a metallic plate (thick grey line), free to rotate about two torsional rods (black dot), subjects to the momentum generated by the nonlinear Casimir force, which arises from the interaction with a fixed metallic sphere of radius R placed at a distance z. The oscillator is excited by the application of a voltage to an electrode fixed under the plate. The choice of the spherical shape for one of the interacting surfaces is justified to avoid alignment problems.

For this arrangement, the Casimir force takes the value [2]

$$F_C = \frac{\pi^3 \hbar c R}{360 z^3} \tag{1}$$

where $\hbar$ is the Planck constant$/2\pi$, and c is the speed of light.

So far, we ignore the dissipation and the forcing. At the equilibrium distance $z = d$, the momentum generated by the Casimir force $M_C = F_C b$ is balanced by the restoring elastic torque $M_e = -\alpha \theta$, inducing a rotation $\theta = \theta_0$,

$$-\alpha \theta_0 + \frac{\pi^3 \hbar c R b}{360 d^3} = 0 \tag{2}$$

[1] Actually the zero-point energy

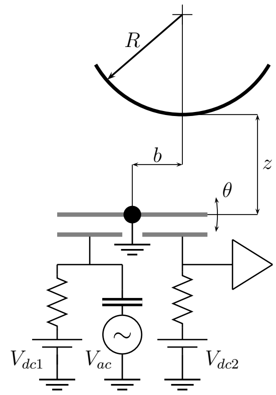

Figure 1: The Casimir nonlinear oscillator [7].

where α is the torsional spring constant. For small oscillations, $z \sim d - b\theta$, and following [7], the Casimir force $F_C(z)$ is Taylor expanded about d up to z^3. The potential energy of the system can be easily calculated giving

$$\begin{aligned} V(\theta) &= \frac{\alpha(\theta+\theta_0)^2}{2} - F_C(d)\, b\,\theta + \frac{F_C'(d)b^2}{2}\,\theta^2 \\ &\quad - \frac{F_C''(d)b^3}{6}\,\theta^3 + \frac{F_C'''(d)b^4}{24}\,\theta^4 \end{aligned} \quad (3)$$

where $F_C(d)$, $F_C'(d)$, $F_C''(d)$, $F_C'''(d)$ are the Casimir force, and its first, second and third derivatives evaluated at a distance d, respectively. The potential $V(\theta)$ has a local minimum in the origin and two local maxima at

$$\begin{aligned} \theta_\pm &= \frac{3}{F_C'''(d)b^4}\left(\frac{F_C''(d)b^3}{2}\right. \\ &\left.\pm\sqrt{\left(\frac{F_C''(d)b^3}{2}\right)^2 - 4\frac{F_C'''(d)b^4}{6}\left(\alpha + F_C'(d)b^2\right)}\right), \end{aligned} \quad (4)$$

and decreases unbounded for both $\theta < \theta_-$ and $\theta > \theta_+$. Therefore the system has a neutrally stable equilibrium in the origin (a center), and two unstable equilibria at $\theta_\pm$, which are of saddle type. The Lagrange equation of motion is

$$\ddot{\theta} = \lambda\,\theta + \mu\,\theta^2 + \nu\,\theta^3 \quad (5)$$

where

$$\lambda = -\left[\omega_0^2 + \frac{F_C'(d)\,b^2}{I}\right];\ \mu = \frac{F_C''(d)\,b^3}{2I};\ \nu = -\frac{F_C'''(d)\,b^4}{6I}. \quad (6)$$

Here, I is the moment of inertia of the plate, and $\omega_0^2 = \alpha/I$ is the fundamental frequency of the oscillator. To simplify eq. (5), we introduce a new variable $\phi = \theta - \theta_+$, and rewrite (5) as a system of first order ODEs

$$\begin{cases} \dot{\phi} &= J \\ \dot{J} &= \nu\,\phi\,(\phi+\theta_+)\,(\phi+\theta_+-\theta_-)\,. \end{cases} \quad (7)$$

Then we introduce new parameters

$$\rho = \nu(2\theta_+ - \theta_-); \qquad \sigma = \nu\,\theta_+(\theta_+ - \theta_-) \quad (8)$$

and perform a linear change of coordinates

$$(\phi, J, t) \to \left(\sqrt{\frac{\sigma}{\nu}}\,\phi,\ \frac{\sigma}{\sqrt{\nu}}J,\ \frac{1}{\sqrt{\sigma}}\,t\right), \quad (9)$$

which reduces eq. (7) to

$$\begin{cases} \dot{\phi} &= J \\ \dot{J} &= \phi^3 + \xi\,\phi^2 + \phi \end{cases} \quad (10)$$

where $\xi = \rho/\sqrt{\nu\,\sigma}$.
Eq. (10) presents the advantage to depend on the parameter ξ only, which adsorbs the three previously defined parameters λ, μ, ν. Moreover, we have shifted one saddle to the origin, the other to ϕ_- and the center to ϕ_+, where

$$\phi_\pm = \frac{1}{2}\left(-\xi \pm \sqrt{\xi^2 - 4}\right). \quad (11)$$

3. Homoclinic Orbit

In this section we show that, for certain values of the parameter ξ, system (10) possesses a homoclinic orbit through the origin, surrounding a region filled with periodic orbits. System (10) has the Hamiltonian

$$H(\phi, J) = \frac{J^2}{2} - \frac{\phi^4}{4} - \frac{\xi\,\phi^3}{3} - \frac{\phi^2}{2}, \quad (12)$$

the level sets $H(\phi, J) = E$ define the trajectories of (10). For the orbit passing through the origin we have $H(0,0) = 0$, which implies

$$J = \pm\sqrt{\phi^2\left(\frac{\phi^2}{2} + \frac{2\xi\,\phi}{3} + 1\right)}. \quad (13)$$

This curve intersects the ϕ–axis in three points, $\tilde{\phi} = 0$, and

$$\tilde{\phi}_\pm = -\frac{2\xi}{3} \pm \sqrt{\frac{4\xi^2}{9} - 2} \quad (14)$$

provided $\xi > 3\sqrt{2}/2$. Introducing the positive determination of (13) in the first of (10) we obtain

$$\dot{\phi} = \sqrt{\phi^2\left(\frac{\phi^2}{2} + \frac{2\xi\,\phi}{3} + 1\right)}. \quad (15)$$

By separation of variables, eq. (15) can be integrated from 0 to t in terms of elementary functions, since it has repeated roots. Choosing the initial conditions as $(\phi(0), J(0)) = (\tilde{\phi}_+, 0)$ we have

$$\phi(t) = 3\,\frac{\tanh^2\frac{t-K}{2} - 1}{2\xi - 3\sqrt{2}\tanh\frac{t-K}{2}} \quad (16)$$

where $K = 2\,\text{atanh}\left(\frac{\sqrt{2}}{2}\tilde{\phi}_+\right)$. Computing the derivative and with some algebraic manipulations we obtain

$$J(t) = \frac{12\xi \sinh(t-K) - 18\sqrt{2}\cosh(t-K)}{\left(2\xi + 2\xi\cosh(t-K) - 3\sqrt{2}\sinh(t-K)\right)^2}. \quad (17)$$

It is readily seen that

$$\lim_{t\to\pm\infty} (\phi(t), J(t)) = (0,0) \quad (18)$$

which matches the requirement for $(\phi(t), J(t))$ to be a homoclinic loop through the origin.

Next we observe that for $\xi > 3\sqrt{2}/2$, eqs (11) and (14) imply $\tilde{\phi}_+ < \phi_+$, that is, the center lies inside the region delimited by the homoclinic orbit. Since outside the interval between the two saddles the potential decreases unbounded, we conclude that the homoclinic orbit is the separatrix between a region filled of periodic orbits and a region characterized by unbounded trajectories.

4. Homoclinic chaos and Melnikov method

Now we include weak dissipation and periodic forcing. The equation of motion becomes

$$\left\{ \begin{array}{rcl} \dot{\phi} &=& J \\ \dot{J} &=& \phi^3 + \xi\phi^2 + \phi + \epsilon\left(A\cos\omega t - \gamma J\right), \end{array} \right. \quad (19)$$

where $\epsilon \ll 1$ takes into account the weakness of the perturbation, A and ω are the amplitude and frequency of the forcing, respectively, and γ is the damping constant. Under the effect of the perturbation, the stable and unstable manifolds split, and may eventually intersect each other transversally, giving rise to transverse homoclinic orbits. The existence of such orbits implies, via the Smale–Birkhoff theorem, the presence of Smale horseshoes, and it is a signature of chaotic behavior [8].

These transversal intersections may be found by searching the simple zeros of the Melnikov function [8]. For the case under investigation the Melnikov function is given by

$$M(t_0) = \int_{-\infty}^{+\infty} J(t)\left[A\cos\omega(t+t_0) - \gamma\, J(t)\right] dt. \quad (20)$$

We split this integrals into two parts. Using integration by parts

$$\int_{-\infty}^{+\infty} J(t)\cos\omega(t+t_0)\,dt = \phi(t)\cos\omega(t+t_0)\Big|_{-\infty}^{+\infty} + \omega\int_{-\infty}^{+\infty}\phi(t)\sin\omega(t+t_0)\,dt. \quad (21)$$

It is easy to see that the first contribution is null, as $\phi(t)$ goes to zero for $t \to \pm\infty$. With the substitution $x = (t-K)/2$, and using the properties of hyperbolic functions, the second contribution becomes

$$\omega\int_{-\infty}^{+\infty}\phi(t)\sin\omega(t+t_0)\,dt = 6\,\omega\int_{-\infty}^{+\infty}\frac{\sin(2x+K+t_0)}{\cosh x\,(3\sqrt{2}\sinh x - 2\xi\cosh x)}\,dx. \quad (22)$$

This integral can be solved by the method of residues, observing that we have regularly spaced simple poles at $z = i(\pi/2 + k\pi)$ and $z = \text{atanh}\,(3\sqrt{2}/(2\xi)) + i(\pi/2 + k\pi)$, and considering the integration path shown in figure 2.

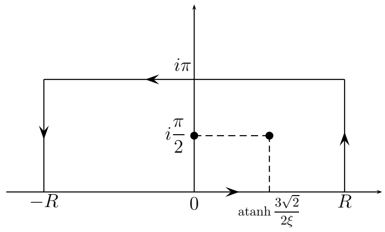

Figure 2: The path of integration for (22).

By using Green theorem, the second part can be recast as a line integral, since it is not explicitly time dependent, obtaining

$$\int_{-\infty}^{+\infty} J^2(t)\,dt = \int_{\Gamma_0} J(\phi)\,d\phi = 2\int_{\tilde{\phi}_+}^{0}\phi\sqrt{\frac{\phi^2}{2} + \frac{2\xi\phi}{3} + 1}\,d\phi, \quad (23)$$

which can be solved by usual methods.

The final results for (20) is

$$M(t_0) = -\gamma\rho + A\,\sigma\,\sin\left[\omega\left(t_0 + K + \text{atanh}\,\frac{3\sqrt{2}}{2\xi}\right)\right] \quad (24)$$

where

$$\rho = 4\frac{3-\xi^2}{9} + \frac{2\xi(4\xi^2-18)}{27\sqrt{2}}\ln\frac{2\xi + 3\sqrt{2}}{\sqrt{4\xi^2-18}} \quad (25)$$

$$\sigma = 2\sqrt{2}\,\pi\,\omega\,\text{cosech}\,(\omega\pi)\,\sin\left(\omega\,\text{atanh}\,\frac{3\sqrt{2}}{2\xi}\right). \quad (26)$$

From (24), the Melnikov function has infinitely many zeros provided

$$\sin\left[\omega\left(t_0 + K + \text{atanh}\,\frac{3\sqrt{2}}{2\xi}\right)\right] = \frac{\rho}{\sigma}\frac{\gamma}{A} \quad (27)$$

These zeros are simple if $\frac{dM(t_0)}{dt_0} \neq 0$. For the derivative we have

$$\frac{dM(t_0)}{dt_0} = \omega A\,\sigma\cos\left[\omega\left(t_0 + K + \text{atanh}\,\frac{3\sqrt{2}}{2\xi}\right)\right]. \quad (28)$$

A sufficient condition for $\frac{dM(t_0)}{dt_0} \neq 0$ when $M(t_0) = 0$ is

$$-1 < \sin\left[\omega\left(t_0 + K + \text{atanh}\,\frac{3\sqrt{2}}{2\xi}\right)\right] < 1 \quad (29)$$

from which we finally derive the condition to have homoclinic tangencies

$$\frac{A}{\gamma} > \left|\frac{\rho}{\sigma}\right|. \tag{30}$$

In figure 3 are shown the stable and unstable manifolds, obtained through numerical simulations, for $\xi = 4$, $\epsilon\gamma = 0.1$, $\omega = 1$, and different values of ϵA. The homoclinic loop is also shown (dashed line) for reference. For A less than the critical value, the manifolds are well apart. For $A = 0.02126$, very close to the theoretical value $A_c = 0.02$ obtained from (30), the first homoclinic tangency occurs. For higher values of A the manifolds intersect transversally.

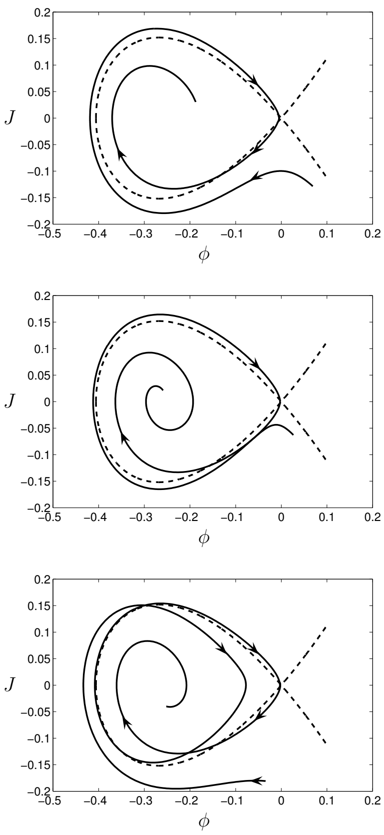

Figure 3: Numerically obtained stable and unstable manifolds for different forcing amplitudes. Upper: $A = 0.017$. Middle: $A = 0.02126$. Lower: $A = 0.025$.

Under the effect of the perturbations, we expect that the periodic trajectories inside the homoclinic loop undergo some kind of bifurcation. In particular, we expect the emergence of limit cycles from the center, with period multiple to that of the periodic forcing. These scenario is commonly know as subharmonic resonances [8], its analysis is left to a future work.

5. Conclusions

Recently, there has been a great amount of attention toward the possible application of QED effects in nano–electro–mechanical devices.

We have analyzed the dynamical behavior of one such apparatus, e.g. the Casimir nonlinear oscillator. Resorting to the Melnikov method, we have shown that, due to the nonlinear nature of Casimir force, when a periodic forcing is applied, the oscillator can exhibit transversal homoclinic intersections between stable and unstable manifolds. Via the Smale–Birkhoff theorem, this implies the existence of Smale's horseshoes.

The importance of this result is twofold. On the one hand, it is relevant in view of possible applications of such oscillator. On the other hand, we have shown that chaotic behavior can arise in a practical device, due to a QED effect. To the best of our knowledge, this result is here reported for the first time.

Acknowledgments

The first author grateful thanks the *Istituto Superiore Mario Boella* and the regional government of Piedmont for financial support. This work was partially supported by the CRT Foundation.

References

[1] H. B. G. Casimir, "On the attraction between two perfectly conducting plates," *Proc. K. Ned. Akad. Wet.*, vol. 60, pp. 793–795, 1948.

[2] P. W. Milonni, *The Quantum Vacuum: An Introduction to Quantum Electrodynamics*. San Diego: Academic Press, 1994.

[3] A. N. Cleland and M. L. Roukes, "A nanometer-scale mechanical electrometer," *Nature*, vol. 392, p. 160-162, 1998.

[4] M. D. LaHaye, O. Buu, B. Camarota, and K. C. Schwab, "Approaching the quantum limit of a nanomechanical resonator," *Science*, vol. 304, pp. 74-77, 2004.

[5] F. M. Serry, D. Walliser and G. J. Maclay, "The anharmonic Casimir oscillator (ACO)–The Casimir effect in a model microelectromechanical system," *Journal of microelectromechanical systems*, vol. 4, no.4, pp. 193–205.

[6] H. B. Chan, V. A. Aksyuk, R. N. Kleinman, D. J. Bishop, and F. Capasso, "Quantum mechanical actuation of microelectromechanical systems by the Casimir force," *Science*, vol. 291, pp. 1941-1944, 2001.

[7] H. B. Chan, V. A. Aksyuk, R. N. Kleinman, D. J. Bishop, and F. Capasso, "Nonlinear micromechanical Casimir oscillator," *Physical Review Letters*, vol. 87, pp. 211 801–1-211 801–4, 2001.

[8] J. Guckenheimer and P. Holmes, "Nonlinear oscillations, dynamical systems and bifurcations of vector fields," 5th edition, Springer-Verlag, New York, 1997.