2010 International Symposium on Nonlinear Theory and its Applications
NOLTA2010, Krakow, Poland, September 5-8, 2010

A constructive approach for the design of finite time self-synchronizing coupled systems with unknown inputs

Gilles Millérioux†

†Nancy University
Research Center for Automatic Control of Nancy (CRAN UMR 7039)
2 rue Jean Lamour, 54519 Vandoeuvre-les-Nancy, France
Email: gilles.millerioux@esstin.uhp-nancy.fr

Abstract—In this paper, we are interested in the problem of synchronization of coupled dynamical systems. The coupling under consideration is unidirectional and corresponds to a drive-response configuration. The drive system is supposed to be subjected to unknown inputs. It is provided a systematic methodology for selecting suitable drive variables and for designing an appropriate response system so that a finite time self-synchronizing is achieved. The approach is based on the notion of flatness, a notion borrowed from control theory.

1. Introduction

Driving a dynamical system has been an important subject of research for a long time. Driving a system by another means that both systems are coupled so that the behavior of the second one is dependent on the behavior of the first one but the converse does not hold. The first system is called the *drive* system while the second one is called the *response*. The driving is often referred to as *unidirectional coupling* and distinguishes from the bidirectional coupling. The coupling is made through the drive variables which consist of one or several outputs variables of the drive system. In the years 1990, works of Hubler [6] have been shown that driving systems with aperiodic signals could induce some interesting behaviors like nonlinear resonances or stimulation of particular modes. The idea has been extended to chaotic signals and originates from the pioneering works of Pecora and Carroll [5].

Among numerous definitions of synchronization (see [2] for an complete list), self-synchronization in a drive-response configuration has drawn much attention. By self-synchronization, it is meant an identical behavior of the drive and the response which is achieved without any external control. The main issue in self-synchronization is not only the selection of appropriate outputs of the drive to guarantee a given convergence behavior as asymptotical, finite-time, robustness against parameter mismatch or disturbances, as well as the design of a suitable structure for the response. Several examples and different situations can be borrowed from the engineering area. For instance, we can be interested in synchronizing two oscillators for communication purpose. The drive may consist of a modulator in a communication setup while the response may be a PLL (Phase Locked-Loop). In such a case, the drive signal is imposed while the structure of the response must be suitable designed to guarantee the phase synchronization with good filtering properties. Another example concerns symmetric cryptography. In such a context, the drive consists of the generator delivering a complex sequence used to conceal the information called the plaintext. The response consists of the decipher which not only must be synchronized in finite-time with the drive but also must be designed so that the plaintext can be properly recovered. The drive variable is nothing else but the ciphertext which is conveyed through the public channel. For a typical class of ciphers, the synchronization must be guaranteed without external control on the decipher. Indeed synchronization flags may be forbidden for throughput purpose. In other words, finite-time self-synchronization must be ensured.

Numerous techniques proposed so far in the literature to guarantee self-synchronization of autonomous dynamical systems resort to state reconstruction approaches involving for example observers. In such a case, the corresponding required property to guarantee synchronization is observability. A more complex self-synchronization issue arises when the drive system is non autonomous, that is forced by an input, and when such an input is unknown to the response. In such a case, a so-called unknown input observer must be used. It is the typical situation encountered in the aforementioned symmetric ciphers where the plaintext plays the role of the unknown input. It is also the case when a drive system is subjected to unknown disturbances. It turns out that unknown input finite-time self-synchronization is an issue which has not been deeply addressed.

In this paper, we propose a methodology leading to a systematic and constructive design of finite time self-synchronizing coupled systems with unknown inputs. Both issues, namely, the selection of suitable drive (output) variables and the design of the response system, are investigated. Analysis approaches have already been suggested in the literature. By analysis, it is meant that given a specific output, it is possible to check whether the finite-time self-synchronization can be achieved. For example, a condition which applies for switched discrete-time systems has been proposed in [4]. On the other hand, design purpose, that is the issue of selecting a priori suitable outputs to achieve finite-time self-synchronization is a much more intricate problem. Actually it is an open problem in the general case. Very few works have addressed such an issue. See the work [3] for an exception dealing with continuous linear systems and a polynomial matrices-based approach. Based on the notion of flatness, a notion borrowed from control theory, we propose here a state space approach for discrete-time linear systems with the hope that an extension can be carried out for some classes of nonlinear systems.

The outline of this paper is the following. In Section 2, the problem of finite-time self-synchronization is stated in the general case. In Section 3 the problems of the selection of appropriate drive variable and of the design of the response which must ensure a finite-time self-synchronization are solved for discrete-time linear systems. An illustrative example is provided in Section 4. Finally Section 5 is devoted to some concluding remarks addressing the possible extension to nonlinear systems.

2. Problem statement in the general case

We are interested in a drive-response setup where the drive part is described by

$$\begin{cases} x_{k+1} = f(x_k, m_k) \\ y_k = h(x_k, m_k) \end{cases} \quad (1)$$

m_k is the input, f is the state-transition function, h is the output function and y_k is the drive (output) variable ensuring the coupling with the response.
The response part admits the following generic equations

$$\begin{cases} \hat{x}_{k+r+1} = \tilde{f}(\hat{x}_{k+r}, y_{k-l}, \ldots, y_{k-l'}) \\ \hat{m}_{k+r} = \tilde{h}(\hat{x}_{k+r}, y_{k-l}, \ldots, y_{k-l'}) \end{cases} \quad (2)$$

where l and l' are integers and where $\tilde{h}$ must have the following property:

$$\hat{m}_{k+r} = m_k \text{ if } \hat{x}_{k+r} = x_k \quad (3)$$

r is a positive integer which stands for a possible delay. The equation involving $\tilde{h}$ plays a central role for the recovery of the input m_k of (1), m_k being assumed to be unknown.

Definition 1 *A finite time self-synchronization fulfills*

$$\exists k_f < \infty, \ \forall \hat{x}_0 \in U, \forall k > k_f \ \textit{and} \ \forall m_k \ \|x_k - \hat{x}_{k+r}\| = 0 \quad (4)$$

where U is a non empty set of initial conditions and $\|\cdot\|$ denotes the Euclidean norm.
Firstly, since the coupling is only unidirectional - from the drive to the response - x_k cannot depend on $\hat{x}_k$. As a result, for all $k > k_f$, when (4) applies, that is when x_k and $\hat{x}_{k+r}$ are equal after a finite number of iterations, the consideration of (2) leads to the fact that x_k and $\hat{x}_k$, up to a delay r, are both expressed as a function, denoted F, which depends exclusively on a finite number of delayed outputs y_k, that is

$$x_k = \hat{x}_{k+r} = F(y_{k-M}, \ldots, y_{k-M'}) \ \ \forall \ k > k_f \quad (5)$$

where M and M' are integers.
Besides, after substituting the expression of $\hat{x}_{k+r}$ into the second equation of (2) and taking into account (3), it turns out that m_k and $\hat{m}_{k+r}$ are equal and both of them can be expressed as a function, denoted G, which depends also exclusively on a finite number of delayed outputs y_k, that is

$$m_k = \hat{m}_{k+r} = G(y_{k-N}, \ldots, y_{k-N'}) \ \ \forall \ k > k_f \quad (6)$$

where N and N' are integers. Let us point out that (6) is nothing else but the input/output model of (1). Such a relation provides a way to recover the unknown input m_k.
The property that the state vector x_k and the input m_k of the dynamical system (1) can be expressed exclusively as a function of the delayed outputs is called flatness. For more details about flatness, the reader may refer to [1]. It is important stressing that, likewise observability, all dynamical systems haven't got this property. The output y_k corresponding to a suitable function h which yields the relations (5) for x_k and (6) for m_k is called the flat output.

Remark 1 *The relation (5) reflects that a flat system is necessary observable insofar as the state vector x_k is expressed by means of a function of the output only.*

Equation (5) and (6) allow to rewrite the response (2) in the strictly equivalent form whenever $k > k_f$

$$\begin{cases} \hat{x}_{k+r} = F(y_{k-M}, \ldots, y_{k-M'}) \\ \hat{m}_{k+r} = G(y_{k-N}, \ldots, y_{k-N'}) \end{cases} \quad (7)$$

The previous developments allows us to state the following proposition.

Proposition 1 *If a dynamical system at the drive side is flat, it is always possible to select an output y_k called flat output and to design a response system so that, not only a self-synchronization in finite time is achieved, but also so that the unknown input m_k of (1) can be recovered in finite time. The equations describing the response are given by (7) or by (2) for a recursive equivalent form.*

The purpose of this paper is to provide a systematic methodology based on a state space approach to select the flat outputs and to design the response system for the special class of discrete-time linear systems with the hope that an extension can be carried out for some classes of nonlinear systems.

3. Main result

3.1. Background on control theory

Throughout this section, when classical linear control theory results are mentioned, proofs are not incorporated. Let us consider the state space representation of a Single Input Single Output linear system:

$$\left\{\begin{array}{lcl} x_{k+1} & = & Ax_k + Bm_k \\ y_k & = & Cx_k + Dm_k \end{array}\right. \tag{8}$$

with $x_k \in \mathbb{R}^n$, $m_k \in \mathbb{R}$ and $y_k \in \mathbb{R}$.
The corresponding input/output model of (8) reads:

$$y_{k+n} + \ldots + a_1 y_{+1} + a_0 y_k = \beta_n m_{k+n} + \ldots + \beta_1 m_{k+1} + \beta_0 m_k \tag{9}$$

where the a_i's are the coefficients of the characteristic polynomial of A which is by definition $\varphi(\lambda) = det(\lambda I - A)$ (I stands for the identity matrix of dimension n and det is the determinant). Equation (9) can be obtained by working out the transfer function $H(z) = Y(z)/M(z)$ of (8) which is given by

$$H(z) = C(z\mathbf{1} - A)^{-1}B + D = \frac{\beta_n z^n + \ldots + \beta_1 z + \beta_0}{z^n + \ldots + a_1 z + a_0} \tag{10}$$

then considering z as the shift operator in the time domain.

Proposition 2 *The system (8) is observable if and only if rank $Q_o = n$ with*

$$Q_o = [C^T \ (CA)^T \ \ldots \ (CA^{n-1})^T]^T$$

Proposition 3 *[1] The system (8) is flat if and only if it controllable, that is, rank $Q_c = n$ with*

$$Q_c = [B \, AB \ \ldots \ A^{n-1}B]$$

3.2. Selecting a flat output for the drive

We wish to state a condition on the space space model (8) which guarantees that y_k is a flat output.

Proposition 4 *The output y_k of (8) is flat whenever the pair (C, D) fulfills*

$$D = D^*, \ C = C^* T^{-1} \text{ with } T \text{ solution of}$$

$$\left\{\begin{array}{l} TA^* = AT \\ B = TB^* \end{array}\right. \tag{11}$$

with

$$A^* = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & \cdots & 0 \end{bmatrix}, B^* = \begin{bmatrix} \beta_{n-1} - \beta_n a_{n-1} \\ \vdots \\ \beta_i - \beta_n a_i \\ \vdots \\ \beta_0 - \beta_n a_0 \end{bmatrix}$$

$$C^* = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad D^* = \beta_n \tag{12}$$

and all the β_is are zero but one denoted β_p, $p \in \{0, \ldots, n\}$

Proof
Firstly, the observable canonical form

$$\left\{\begin{array}{lcl} x^*_{k+1} & = & A^* x^*_k + B^* m_k \\ y_k & = & C^* x^*_k + D^* m_k \end{array}\right. \tag{13}$$

with A^*, B^*, C^* and D^* defined by (12) and the state space representation (8), related one another according to

$$\begin{array}{l} A^* = T^{-1}AT \\ B^* = T^{-1}B \\ C^* = CT \\ D^* = D \end{array} \tag{14}$$

from which (11) are deduced, have the same input/output model (9). T is the similarity transform matrix and is invertible by definition. To prove such a correspondence, it suffices to work out the transfer function $H^*(z) = C^*(z\mathbf{1} - A^*)^{-1}B^* + D^*$ to realize that is the same as (10) and then considering again the variable z as a shift operator.
Since it is assumed that only one term β_i with $i = p$ ($p \in \{0, \ldots, n\}$) is different from zero, (9) reduces to

$$\sum_{j=0}^{n-1} a_j y_{k+j} + y_{k+n} = \beta_p m_{k+p} \tag{15}$$

Consequently, the first condition for a flat output, that is the input must be expressed as a finite number of delayed outputs y_k, is fulfilled.
Secondly, let us iterate (8) and lump together the iterates in the following matrix form

$$\begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+n-1} \end{bmatrix} - \Gamma \begin{bmatrix} m_k \\ m_{k+1} \\ \vdots \\ m_{k+n-1} \end{bmatrix} - Q_o x_k = 0 \tag{16}$$

with

$$\Gamma = \begin{bmatrix} D & 0 & \ldots & 0 \\ CB & D & \vdots & 0 \\ \vdots & \ldots & \vdots & 0 \\ CA^{n-1}B & \ldots & \vdots & D \end{bmatrix}, \qquad Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Notice that the matrix Q_o is invertible since by construction, (8) is observable and precisely, Q_0 is the observability matrix of (8). As a result, one gets

$$x_k = Q_0^{-1}\left(\begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+n-1} \end{bmatrix} - \Gamma \begin{bmatrix} m_k \\ m_{k+1} \\ \vdots \\ m_{k+n-1} \end{bmatrix} \right) \tag{17}$$

Since m_k and its iterates depend exclusively on a finite number of delayed outputs y_k regarding (15), so does the state vector x_k of (17). As a consequence, the second condition required for a flat output is fulfilled. That completes the proof.

Based on Proposition 1, several important remarks can be made.

Remark 2 *The tractability of this result lies in that the unknown T, which enables to compute C, can be easily obtained since (11) are mere linear matrix equalities to be solved.*

Remark 3 *Given A^* and B^*, the solution T of (11) is not unique*

Remark 4 *The whole uncountable set of flat outputs of (8) corresponds to the set of all triplets (p, β_p, T) with $p \in \{1, \ldots, n\}, \beta_p \in \mathbb{R}$ and T solutions of (11)*

3.3. Design of the response

On one hand, from (15), we infer that the function G of (7) fulfills:

$$\begin{array}{rcl} \hat{m}_{k+r} & = & m_k = \beta_p^{-1} \cdot (\sum_{j=0}^{n-1} a_j y_{k-p+j} + y_{k-p+n}) \\ & = & G(y_{k-N}, \ldots, y_{k-N'}) \end{array} \quad (18)$$

where r is the delay introduced for causality sake.
Next, substituting m_k given by (18) and its iterates into (17) and replacing x_k by $\hat{x}_{k+r}$ gives explicitly the function F of (7).

4. Example

We consider the following drive system with state space model

$$\left\{ \begin{array}{rcl} x_{k+1} & = & \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ -1 \end{bmatrix} m_k \\ y_k & = & C x_k + D m_k \end{array} \right. \quad (19)$$

We aim to find an appropriate output y_k and so suitable matrices C and D as well as designing a response system so that the resulting drive-response configuration has the finite-time self-synchronization property.

The controllability matrix Q_c is

$$Q_c = \begin{bmatrix} 1 & -1 \\ -1 & -4 \end{bmatrix}$$

The system is controllable since rank $Q_c = n = 2$. According to Proposition 3, it is flat. As a result, we can check for a flat output y_k.
The characteristic polynomial of A is $\varphi(\lambda) = det(\lambda I - A) = \lambda^2 - 4\lambda + 5$. Thus, the matrix A^* of the observable canonical form reads

$$A^* = \begin{bmatrix} 4 & 1 \\ -5 & 0 \end{bmatrix} \quad (20)$$

We set arbitrarily $p = 1$ and $\beta_1 = 1$. It is recalled that according to Proposition 4, all the other coefficients β_i with $i \neq p$, here β_0 and β_2, must be zero. Thus B^* reads

$$B^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (21)$$

To find out C and D which ensures a flat output, we must consequently solve (11). One gets

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = [0 \ -1], \quad D = 0$$

Following Section 3.3, the corresponding response system which ensures a finite-time self-synchronization with C and D previously obtained yields, after basic matrices manipulations

$$\hat{m}_{k+1} = y_{k+1} - 4y_k + 5y_{k-1}$$
$$\hat{x}_{k+1} = \begin{pmatrix} y_k - 5y_{k-1} \\ -y_k \end{pmatrix}$$

Let us notice that a delay $r = 1$ has been introduced for causality sake.

5. Concluding remarks

In this paper, we have provided a systematic methodology for achieving a finite time self-synchronization between two unidirectional coupled systems. Both issues, namely the selection of suitable drive variables and the design of an appropriate response system, have been addressed. The approach is based on state space models and the notion of flatness. Whether the method can be extended to nonlinear systems is an interesting but difficult matter. Indeed, the key idea of the present paper lies in that we are able to find out an equivalence between two objects: a general state space model on one hand and a canonical state space model on the other hand. Owing to the equivalence, the property of flatness for the canonical form induces the same property for the general state space model. The trick lies in that characterizing flatness for the canonical form is straightforward. As a result, if we wish to extend the approach for nonlinear systems, we must find out canonical forms, also called normal forms, characterizing flat systems. Such an issue deserves further works.

References

[1] M. Fliess, J. Levine, P. Martin, and P. Rouchon. Flatness and defect of non-linear systems: introductory theory and examples. *Int. Jour. of Control*, 61(6):1327–1361, 1995.

[2] Blekhman I.I., Fradkov A.L., Nijmeijer H., and Pogromsky A.Y. On self-synchronization and controlled synchronization. *Systems and Control letters*, 31(5):299–305, 1997.

[3] J. Levine and D. V. Nguyen. Flat output characterization for linear systems using polynomial matrices. *Systems and control Letters*, 48:69–75, 2003.

[4] G. Millérioux and J. Daafouz. Flatness of switched linear discrete-time systems. *IEEE Trans. on Automatic Control*, 54(3):615–619, March 2009.

[5] L.M. Pecora and T.L. Carroll. Synchronization in chaotic systems. *Phys. Rev. Lett.*, 64:821–824, 1990.

[6] B. B. Plapp and A. W. Hübler. *Physical Review Letters*, 65(232), 1990.