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Tensor rank determination problem

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Abstract—The determination of the maximal rank of a set of a given type of tensors is a basic problem both in theory and application. In fact the problem has been treated in various statistical and mathematical fields. In this talk we first review our recent results about the maximal rank of tensors with two slices and three slices, and secondly talk about our new proof of "a generic Kronecker form" and finally introduce the connection to quiver representation.

1. Introduction

Tensor is another name of high dimensional array of data. Recently we have witnessed many applications of tensor data in broad field such as brain wave analysis, image analysis, web analysis and more.

Given a k-dimensional tensor $T = (t_{i_1 i_2 \cdots i_k})$ of size $n_1 \times \cdots \times n_k$ with entries in a field $\mathbb{K}$, we associate an element $x \in \mathbb{K}^{n_1} \otimes \cdots \otimes \mathbb{K}^{n_k}$ such that $x = \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} t_{i_1 \cdots i_k} \boldsymbol{e}_{i_1} \otimes \cdots \otimes \boldsymbol{e}_{i_k}$, where $\boldsymbol{e}_i$ is the i-th fundamental vector. It is known that x can be expressed as a sum of finite tensors of form $\boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_k$. The smallest number of the tensors of the form $\boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_k$ need to express x as a sum of them is called the rank of x. In terms of high dimensional array datum, $T = (t_{i_1 \cdots i_k}) \neq O$ is rank one if and only if there are vectors $\boldsymbol{t}^{(j)} = (t_1^{(j)}, \ldots, t_{n_j}^{(j)})^T$ $(j = 1, \ldots, k)$ such that $t_{i_1 \cdots i_k} = \prod_{j=1}^{k} t_{i_j}^{(j)}$. Therefore the rank of a high dimensional array datum is the smallest number of high dimensional data of simplest form that generate it as a sum.

So it is worth to study the maximal rank of tensors of given size. And it also important to know the ranks which appear with positive probability if the entries of a tensor with fixed size varies randomly. These ranks are called the typical ranks. See for example [5] and [6].

In this paper, we classify the typical ranks of preassigned size of 3-dimensional tensor in connection with the canonical form of the tensor. Relation to the representation theory of quivers is also discussed.

2. Preliminary

We first recall some basic facts and set terminology.

Notation

1. We denote by $\mathbb{K}$ an arbitrary field and by $\mathbb{F}$, the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$.
2. We denote by E_n the $n \times n$ identity matrix.
3. For a tensor $x \in \mathbb{K}^m \otimes \mathbb{K}^n \otimes \mathbb{K}^p$ with $x = \sum_{ijk} a_{ijk} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$, we identify x with $(A_1; \cdots; A_p)$, where $A_k = (a_{ijk})_{1 \le i \le m, 1 \le j \le n}$ for $k = 1, \ldots, p$ is an $m \times n$ matrix, and call $(A_1; \cdots; A_p)$ a tensor.
4. We denote the set of $m \times n \times \ell$ tensors by $\mathbb{K}^{m \times n \times \ell}$.
5. For an $m \times n \times p$ tensor $T = (A_1; \cdots; A_p)$, an $l \times m$ matrix P and an $n \times k$ matrix Q, we denote by PTQ the $l \times k \times p$ tensor $(PA_1Q; \cdots; PA_pQ)$.
6. For p $m \times n$ matrices $A_1, \ldots, A_p$, we denote by $(A_1, \ldots, A_p)$ the $m \times np$ matrix obtained by aligning $A_1, \ldots, A_p$ horizontally.
7. We set $A_1 \oplus A_2 \oplus \ldots \oplus A_t = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_t \end{pmatrix}$ for matrices $A_1, A_2, \ldots, A_t$ and define $T_1 \oplus T_2 \oplus \ldots \oplus T_t$ similarly for tensors $T_1, T_2, \ldots, T_t$ with ℓ slices.
8. For an $m \times n$ matrix M, we denote the $m \times j$ (resp. $m \times (n-j)$) matrix consisting of the first j (resp. last $n-j$) columns of M by $M_{\le j}$ (resp. ${}_{j<}M$). We denote the $i \times n$ (resp. $(m-i) \times n$) matrix consisting of the first i (resp. last $m-i$) rows of M by $M^{\le i}$ (resp. ${}^{i<}M$).

Definition 1 *Let x be an element of $\mathbb{K}^m \otimes \mathbb{K}^n \otimes \mathbb{K}^p$. We define the rank of x, denoted by* $\operatorname{rank} x$, *to be* $\min\{r \mid \exists \boldsymbol{a}_i \in \mathbb{K}^m, \exists \boldsymbol{b}_i \in \mathbb{K}^n, \exists \boldsymbol{c}_i \in \mathbb{K}^p$ *for $i = 1, \ldots, r$ such that* $x = \sum_{i=1}^{r} \boldsymbol{a}_i \otimes \boldsymbol{b}_i \otimes \boldsymbol{c}_i\}$. $\max\{\operatorname{rank} x \mid x \in \mathbb{K}^m \otimes \mathbb{K}^n \otimes \mathbb{K}^p\}$ *is denoted by* $\max.\operatorname{rank}_{\mathbb{K}}(m, n, p)$.

Definition 2 *Two tensors T and T' are said to be* equivalent *if there are nonsingular matrices P and Q such that $T' = PTQ$.*

Lemma 3 *If T and T' are equivalent, then* $\operatorname{rank} T = \operatorname{rank} T'$.

3. Kronecker canonical forms

We summarize briefly about Kronecker canonical forms.

Lemma 4 ([1, (30) in §4, XII]) *Let A and B be $m \times n$ rectangular matrices. Then $(A; B)$ is equivalent to a tensor of a block diagonal form*

$$(S_1; T_1) \oplus \cdots \oplus (S_r; T_r),$$

where each $(S_j; T_j)$ is one of the following

(A) $k \times \ell \times 2$ *tensor* $(O; O)$,

(B) $k \times k \times 2$ *tensor* $(\lambda E_k + J_k; E_k)$,

(C) $2k \times 2k \times 2$ *tensor* $(C_k(c, s) + J_k \otimes E_2; E_{2k})$, $s \neq 0$,

(D) $k \times k \times 2$ *tensor* $(E_k; J_k)$,

(E) $k \times (k+1) \times 2$ *tensor* $((\mathbf{0}, E_k); (E_k, \mathbf{0}))$,

(F) $(k+1) \times k \times 2$ *tensor* $(\begin{pmatrix} \mathbf{0}^T \\ E_k \end{pmatrix}; \begin{pmatrix} E_k \\ \mathbf{0}^T \end{pmatrix})$.

Here $J_k = \begin{pmatrix} 0 & 1 & & O \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$ *is a $k \times k$ square matrix and* $C_k(c, s) = E_k \otimes \begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \mathrm{Diag}(\begin{pmatrix} c & -s \\ s & c \end{pmatrix}, \ldots, \begin{pmatrix} c & -s \\ s & c \end{pmatrix})$ *is a $2k \times 2k$ square matrix.*

This decomposition is called the Kronecker canonical form. It is unique up to permutations of blocks. Note that tensors of type (A) include ones when $k > 0$ and $\ell = 0$, or $k = 0$ and $\ell > 0$, where a direct sum of a $0 \times \ell \times 2$ tensor of type (A) and an $s \times t \times 2$ tensor $(X; Y)$ means a $k \times (\ell + t) \times 2$ tensor $((O, X); (O, Y))$. Also note that type (C) does not appear over the complex number field $\mathbb{C}$.

Let A and B be $m \times n$ rectangular matrices. We have described the rank of a tensor $(A; B)$ with its Kronecker canonical form. Suppose that the Kronecker canonical form $(S; T)$ of $(A; B)$ has an $m_A \times n_A \times 2$ tensor $(O; O)$ of type (A), ℓ_E tensors of type (E) and ℓ_F tensors of type (F) respectively. And let α be the maximum of the following numbers.

1. $\max_{\lambda \in \mathbb{F}} \beta(\lambda)$, where $\beta(\lambda)$ is the number of blocks of the form $(\lambda E_k + J_k; E_k)$ with $k \geq 2$ appearing in the Kronecker canonical form of $(A; B)$.

2. $\max_{c,s \in \mathbb{R}, s \neq 0} \gamma(c, s)$, where $\gamma(c, s)$ is the number of blocks of the form $(C_k(c, s) + J_k \otimes E_2; E_{2k})$ appearing in the Kronecker canonical form of $(A; B)$.

3. The number of blocks of the form $(E_k; J_k)$ appearing in the Kronecker canonical form of $(A; B)$.

Then

Theorem 5 ([3]) *In the above notation, it holds $m - m_A + \ell_E = n - n_A + \ell_F$ and*

$$\mathrm{rank}_{\mathbb{F}}(A; B) = \alpha + m - m_A + \ell_E \ .$$

4. Generic Kronecker form

In this section, we state another proof of the result of Berge and Kiers [4] of the existence of the "generic Kronecker form" for non-square tensors with 2 slices.

Theorem 6 *Let $\mathbb{K}$ be an infinite field, m, n integers with $0 < m < n$ and $\{x_{ijk}\}_{1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq 2}$ indeterminates over $\mathbb{K}$. Then there is a non-zero polynomial $f(x_{ijk}) \in \mathbb{K}[x_{ijk} \mid 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq 2]$ satisfying the following condition.*

If we set $U = \{T = [t_{ijk}] \in \mathbb{K}^{m \times n \times 2} \mid f(t_{ijk}) \neq 0\}$, then for any $T \in U$, there are nonsingular matrices P and Q, with sizes m and n respectively, such that

$$PTQ = ((O, E_m); (E_m, O)).$$

In the case where $0 < m < n \leq 2m$, $((O, E_m); (E_m, O))$ can be transformed, by row and column permutations, to

$$((\mathbf{0}, E_{q+1}); (E_{q+1}, \mathbf{0}))^{\oplus r} \oplus ((\mathbf{0}, E_q); (E_q, \mathbf{0}))^{\oplus r'},$$

where $q = \lfloor m/(n-m) \rfloor$, $r = m - (n-m)q$ and $r' = n - m - r$. Therefore, we see the following

Corollary 7 *Suppose $K = \mathbb{R}$ or $\mathbb{C}$ and m, n, q, r, r' as above. If $T = [t_{ijk}]$ is an $m \times n \times 2$-tensor whose entries are independent continuous stochastic variables, then the Kronecker canonical form of T is*

$$((\mathbf{0}, E_{q+1}); (E_{q+1}, \mathbf{0}))^{\oplus r} \oplus ((\mathbf{0}, E_q); (E_q, \mathbf{0}))^{\oplus r'}$$

with probability 1. *In particular,* $\mathrm{rank} T = n$ *with probability* 1.

Before we state the proof of Theorem 6, we restate this theorem in the terminology of algebraic geometry.

Theorem 8 *Let $\mathbb{K}$ be an infinite field, m, n integers with $0 < m < n$. Then there are rational maps $\varphi^{(1)}$ and $\varphi^{(2)}$ from $\mathbb{K}^{m \times n \times 2}$ to $\mathrm{GL}(m; \mathbb{K})$ and $\mathrm{GL}(n; \mathbb{K})$ respectively, such that $((O, E_m); (E_m; O))$ is contained in the intersection of the domains of $\varphi^{(1)}$ and $\varphi^{(2)}$ and for any T contained in it*

$$\begin{aligned} \varphi^{(1)}(T) T \varphi^{(2)}(T) &= ((O, E_m); (E_m; O)) \\ \varphi^{(1)}((O, E_m); (E_m; O)) &= E_m \\ \varphi^{(2)}((O, E_m); (E_m; O)) &= E_n. \end{aligned}$$

It is clear that Theorem 6 follows form Theorem 8.

Theorem 8 follows from Lemmas 9, 11 and 12.

Lemma 9 *There is a rational map φ_0 form $\mathbb{K}^{m \times n \times \ell}$ to $\mathrm{GL}(n; \mathbb{K})$ such that $((O, E_m); A_2; \cdots; A_\ell)$ is contained in the domain of φ_0 and*

$$\varphi_0((O, E_m); A_2; \cdots; A_\ell) = E_n$$

for any $m \times n$ matrices $A_2, \ldots, A_\ell$ and

$$(A_1; \cdots; A_\ell)\varphi_0((A_1; \cdots; A_\ell)) = ((O, E_m); *; \cdots; *)$$

for any $(A_1; \cdots; A_\ell)$ in the domain of φ_0.

Proof It is enough to set

$$\varphi_0((A_1; \cdots; A_\ell)) = \begin{pmatrix} A_1 \\ E_{n-m} \; O \end{pmatrix}^{-1} \begin{pmatrix} O & E_m \\ E_{n-m} & O \end{pmatrix}.$$

▮

Lemma 10 *There are rational maps $\varphi_{m\times n,1}$ and $\varphi_{m\times n,2}$ from $\mathbb{K}^{m\times n\times 2}$ to $\mathrm{GL}(m;\mathbb{K})$ and $\mathrm{GL}(n;\mathbb{K})$ respectively, such that $((O, E_m); (E_m, O))$ is contained in the intersection of the domains of $\varphi_{m\times n,1}$ and $\varphi_{m\times n,2}$,*

$$\begin{aligned} \varphi_{m\times n,1}((O, E_m); (E_m, O)) &= E_m \\ \varphi_{m\times n,2}((O, E_m); (E_m, O)) &= E_n \end{aligned}$$

and

$$\begin{aligned} &\varphi_{m\times n,1}((O, E_m); B)((O, E_m); B)\varphi_{m\times n,2}((O, E_m); B) \\ = \;& ((O, E_m); (\boldsymbol{e}_1, *)) \end{aligned}$$

for any $m \times n$ matrix B.

Proof It is enough to set

$$\varphi_{m\times n,1}(A; B) = (B_{\leq m})^{-1}$$

and

$$\varphi_{m\times n,2}(A; B) = \begin{pmatrix} E_{n-m} & O \\ O & B_{\leq m} \end{pmatrix}.$$

▮

Lemma 11 *There are rational maps $\psi_{m\times n,1}$ and $\psi_{m\times n,2}$ from $\mathbb{K}^{m\times n\times 2}$ to $\mathrm{GL}(m;\mathbb{K})$ and $\mathrm{GL}(n;\mathbb{K})$ respectively, such that $((O, E_m); (E_m, O))$ is contained in the intersection of the domains of $\psi_{m\times n,1}$ and $\psi_{m\times n,2}$,*

$$\begin{aligned} \psi_{m\times n,1}((O, E_m); (E_m, O)) &= E_m \\ \psi_{m\times n,2}((O, E_m); (E_m, O)) &= E_n \end{aligned}$$

and

$$\begin{aligned} &\psi_{m\times n,1}((O, E_m); B)((O, E_m); B)\psi_{m\times n,2}((O, E_m); B) \\ = \;& ((O, E_m); (V, *)) \end{aligned}$$

for any $m\times n$ matrix B, where V is an $m\times m$ upper triangular unipotent matrix.

Proof We prove by induction on m. If $m = 1$, we may set $\psi_{m\times n,1} = \varphi_{m\times n,1}$ and $\psi_{m\times n,2} = \varphi_{m\times n,2}$.

Next we assume that $m > 1$ and $\psi_{(m-1)\times(n-1),1}$ and $\psi_{(m-1)\times(n-1),2}$ are defined. We set

$$T' = {}^{1<}_{1<}(\varphi_{m\times n,1}(T)T\varphi_{m\times n,2}(T))$$

for a tensor T contained in the intersection of the domains of $\varphi_{m\times n,1}$ and $\varphi_{m\times n,2}$ and set

$$\psi_{m\times n,1}(T) = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \psi_{(m-1)\times(n-1),1}(T') \end{pmatrix} \varphi_{m\times n,1}(T)$$

and

$$\psi_{m\times n,2}(T) = \varphi_{m\times n,2}(T) \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \psi_{(m-1)\times(n-1),2}(T') \end{pmatrix}$$

for any tensor T which is contained in the intersection of the domains of $\varphi_{m\times n,1}$ and $\varphi_{m\times n,2}$ and T' is contained in the intersection of the domains of $\psi_{(m-1)\times(n-1),1}$ and $\psi_{(m-1)\times(n-1),2}$.

Since $((O, E_m); (E_m, O))$ is contained in the intersection of the domains of $\varphi_{m\times n,1}$ and $\varphi_{m\times n,2}$ and $((O, E_m); (E_m, O))' = ((O, E_{m-1}); (E_{m-1}, O))$, we see that $\psi_{m\times n,1}$ and $\psi_{m\times n,2}$ are defined at $((O, E_m); (E_m, O))$. Therefore, $\psi_{m\times n,1}$ and $\psi_{m\times n,2}$ are rational maps from $\mathbb{K}^{m\times n\times 2}$ to $\mathrm{GL}(m;\mathbb{K})$ and $\mathrm{GL}(n;\mathbb{K})$ respectively whose domains contain $((O, E_m); (E_m, O))$. It is easy to see that

$$\psi_{m\times n,1}((O, E_m); (E_m, O)) = E_m$$

and

$$\psi_{m\times n,2}((O, E_m); (E_m, O)) = E_n.$$

Furthermore,

$$\begin{aligned} &\psi_{m\times n,1}((O, E_m); B)((O, E_m); B)\psi_{m\times n,2}((O, E_m); B) \\ = \;& \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \psi_{(m-1)\times(n-1),1}(((O, E_m); B)') \end{pmatrix} ((O, E_m); (\boldsymbol{e}_1, *)) \\ & \times \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \psi_{(m-1)\times(n-1),2}(((O, E_m); B)') \end{pmatrix} \\ = \;& ((O, E_m); \begin{pmatrix} 1 & * \\ \mathbf{0} & V' \; * \end{pmatrix}) \end{aligned}$$

where V' is an $(m-1)\times(m-1)$ upper triangular unipotent matrix. So $\psi_{m\times n,1}$ and $\psi_{m\times n,2}$ satisfy the required conditions. ▮

Lemma 12 *Let T be an element of $\mathbb{K}^{m\times n\times 2}$ of the form*

$$T = ((O, E_m); (V, M)),$$

where V is an $m\times m$ upper triangular unipotent matrix and M is an $m \times (n-m)$ matrix. Then there are $m \times m$ upper triangular unipotent matrix P and $n \times n$ upper triangular unipotent matrix Q such that

$$PTQ = ((O, E_m); (E_m, O)).$$

Proof We prove by induction on m. Since the case where $m = 1$ is easy, we assume that $m > 1$.

Set $P' = V^{-1}$. Then there is an $n \times n$ matrix Q' of the form

$$Q' = \begin{pmatrix} E_m & * \\ O & E_{n-m} \end{pmatrix}$$

such that

$$P'(V, M)Q' = (E_m, O).$$

Since Q' is an upper triangular unipotent matrix, $P'(O, E_m)Q'$ is of the form (O, V'), where V' is an $m \times m$ upper triangular unipotent matrix.

Set

$$Q'' = \begin{pmatrix} E_{n-m} & O \\ O & (V')^{-1} \end{pmatrix}.$$

Then

$$P'TQ'Q'' = \begin{cases} ((O, E_m); (\begin{pmatrix} E_{n-m} & O \\ O & V'' \end{pmatrix}, O)) & \text{if } n < 2m, \\ ((O, E_m); (E_m, O)) & \text{if } n \geq 2m, \end{cases}$$

where V'' is a $(2m-n)\times(2m-n)$ upper triangular unipotent matrix. So we may put $P = P'$ and $Q = Q'Q''$ in the case where $n \geq 2m$. In the case where $n < 2m$, we see that there are a $(2m - n) \times (2m - n)$ upper triangular unipotent matrix P''' and an $m \times m$ upper triangular unipotent matrix Q''' such that

$$P'''((O, E_{2m-n}); (V'', O))Q''' = ((O, E_{2m-n}); (E_{2m-n}, O))$$

by the induction hypothesis. It is easy to see that we may put

$$P = \begin{pmatrix} E_{n-m} & O \\ O & P''' \end{pmatrix} P'$$

and

$$Q = Q'Q'' \begin{pmatrix} E_{n-m} & O \\ O & Q''' \end{pmatrix}.$$

∎

5. Relation to the representation of quivers

A (finite) quiver is a pair $Q = (Q_0, Q_1)$ of finite sets of vertices Q_0 and arrows Q_1 between the vertices. Formally, Q_0 and Q_1 are disjoint sets and there are maps $s\colon Q_1 \to Q_0$ and $t\colon Q_1 \to Q_0$. We interpret $\rho \in Q_1$ is an arrow from $s(\rho)$ to $t(\rho)$.

A representation X of a quiver Q is a family of vector spaces $\{X_x\}_{x\in Q_0}$ and linear maps $\{f_\rho\}_{\rho\in Q_1}$ such that $f_\rho\colon X_{s(\rho)} \to X_{t(\rho)}$. A morphism between representations of Q is a family $\{F_x\colon X_x \to X'_x\}_{x\in Q_0}$ of linear maps such that $f'_\rho \circ F_{s(\rho)} = F_{t(\rho)} \circ f_\rho$ for any $\rho \in Q_1$.

Since a linear map corresponds to a matrix if one fixes a bases of vector spaces, we may see that a tensor with two slices give a representation of the following quiver.

$$\bullet \rightrightarrows \bullet$$

Tensor may vary by the choice of the bases of vector spaces, but up to equivalence of tensors, there is a one to one correspondence between the set of isomorphism classes of representations of the quiver $\bullet \rightrightarrows \bullet$ and the equivalence classes of tensors with 2 slices.

A principal research area of representation theory is a classification of indecomposable representations. In fact, blocks of types (B) to (F) of the Kronecker canonical form correspond to the indecomposable representation of the quiver $\bullet \rightrightarrows \bullet$.

Now consider tensors with 3 slices. These tensors corresponds to a representation of the following quiver.

$$\bullet \Rrightarrow \bullet$$

It is known that the category of the representations of the above quiver is "wild", i.e., it is so difficult to classify the indecomposable representations that it is hopeless to accomplish.

But there might be some possibility to classify indecomposable representations if one restrict only to the generic case. As a possible first step, we state the following theorem which is proved by using Lemma 9, Theorem 8 and easy calculation.

Theorem 13 *Let $\mathbb{K}$ be an infinite field and m, n integers with $2m < n \leq 3m$. Then there are rational maps $\xi^{(1)}$ and $\xi^{(2)}$ from $\mathbb{K}^{m\times n\times 3}$ to $\mathrm{GL}(m;\mathbb{K})$ and $\mathrm{GL}(n;\mathbb{K})$ respectively, such that*

$$\begin{aligned} & \xi^{(1)}(T)T\xi^{(2)}(T) \\ = \ & ((O, E_m); (O, E_m, O_{m\times m}); (E_m, O_{m\times(n-2m)}, M)) \end{aligned}$$

for any T contained in the intersection of the domains of $\xi^{(1)}$ and $\xi^{(2)}$, where M is an $m\times m$ matrix with $M^{\leq n-2m} = O$.

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