Stock Portfolio Optimization Based on Nonlinear Prediction and DCC-GARCH Model

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Abstract—Two ideas are introduced for the purpose of improving portfolios in terms of both profit and safety. Firstly, a nonlinear prediction technique is applied to estimate the expected return rate, and the DCC-GARCH model is applied to estimate the risk. Secondly, the long and short strategy is considered as an effective way to reduce the risk that is caused by highly correlated stocks. Some investment simulations based on real financial data show that the proposed method is successful in making better portfolios.

1. Introduction

Markowitz's mean-variance portfolio model[1] is well-known as a classical portfolio model, and its portfolio effect can spread the whole risk of a portfolio. However, this portfolio model has at least two problems. Firstly, the estimation of a expected return rate and a risk may not be practical to predict real financial markets. Secondly, in real financial markets, stock movements are often synchronized because of common risk factors. It means that it is difficult to reduce the risk of a portfolio.

To solve the second problem, Meucci proposed the principal-component portfolio model[2], which uses component analysis to make uncorrelated composite stocks for enhancing the portfolio effect. However, this portfolio model gives up the estimation of future expected return rates and risks. Namely, financial markets are considered almost random and difficult to predict their future[3].

However, if we use data-mining techniques like nonlinear prediction models, there is the possibility to predict future moments at least better than conventional portfolio models. So, in the present study, we aggressively use some advanced prediction model in terms of data-mining techniques. First, we apply a nonlinear prediction model to improve the prediction power of future return rates more than Markowitz's portfolio model. Then, we consider its prediction errors as investment risks, and uses the multivariate GARCH model to learn the temporal pattern of risks and to predict their future. Moreover, to solve the second problem, we apply the long and short strategy. It can reduce the whole risk of a portfolio even if stock movements are synchronized with each other. Finally, we perform some investment simulations based on real stock data to confirm the validity of our portfolio strategy.

2. Classical Portfolio Models

We denote $r_i(t)$ as the return rate of ith stock ($i = 1, 2, \cdots, n$) at the time of t. Then, we denote the expected return rate and the risk at $t+1$ as $\hat{r}_i(t+1)$ and $\hat{\sigma}_i^2(t+1)$. According to the original portfolio theory[1], if we make a portfolio by allocation rates $\boldsymbol{d}(t) = [d_1(t), d_2(t), \cdots, d_n(t)]$, the expected return rate $\hat{r}_p(t+1)$ and risk $\hat{\sigma}_p(t+1)$ of a portfolio are respectively estimated by

$$\hat{r}_p(t+1) = \boldsymbol{d}(t)\hat{\boldsymbol{r}}(t+1), \tag{1}$$

$$\hat{\sigma}_p^2(t+1) = \boldsymbol{d}(t)\hat{\boldsymbol{\Sigma}}(t+1)\boldsymbol{d}^{\mathrm{t}}(t), \tag{2}$$

where $\hat{\boldsymbol{r}}(t+1) = [\hat{r}_1(t+1), \hat{r}_2(t+1), \cdots, \hat{r}_n(t+1)]^{\mathrm{t}}$ and $\hat{\boldsymbol{\Sigma}}(t+1)$ is the risk matrix based on each element $\hat{\sigma}_{ij}(t+1)$, which means the expected covariance caused by ith and jth stocks, and $\hat{\sigma}_{ii}(t+1)$ means $\hat{\sigma}_i^2(t+1)$.

2.1. Mean-variance Portfolio Model

In the mean-variance portfolio model[1], the expected return rate $\hat{r}_i(t+1)$ is obtained by the simple moving average of the last T period as follows:

$$\begin{aligned}\hat{r}_i(t+1) &= \bar{r}_i(t)\\ &= \frac{1}{T}\sum_{a=0}^{T-1} r_i(t-a).\end{aligned} \tag{3}$$

Then, the estimated risk is given by the covariance of the last T period:

$$\begin{aligned}\hat{\sigma}_{ij}(t+1) &= \bar{\sigma}_{ij}(t)\\ &= \frac{1}{T}\sum_{a=0}^{T-1}[r_i(t-a)-\bar{r}_i(t)]\cdot\left[r_j(t-a)-\bar{r}_j(t)\right].\end{aligned} \tag{4}$$

By substituting Eqs.(3) and (4) into Eqs.(1) and (2), we can get portfolio's expected return rate $\hat{r}_p(t+1)$ and risk $\hat{\sigma}_p(t+1)$.

Finally, to optimize the allocation rate $\boldsymbol{d}(t)$, we allocate $\boldsymbol{d}(t)$ so as to maximize $\hat{r}_p(t+1)$ and minimize $\hat{\sigma}_p(t+1)$. For this reason, the Sharpe ratio N_{SR}[4] is often used as a typical measure:

$$N_{SR} = \frac{\hat{r}_p(t+1)-r_f(t+1)}{\hat{\sigma}_p(t+1)}, \tag{5}$$

where $r_f(t+1)$ means a risk-free rate, and optimize $\boldsymbol{d}(t)$ so as to maximize N_{SR}. This is a nonlinear programming problem solved by the interior point method.

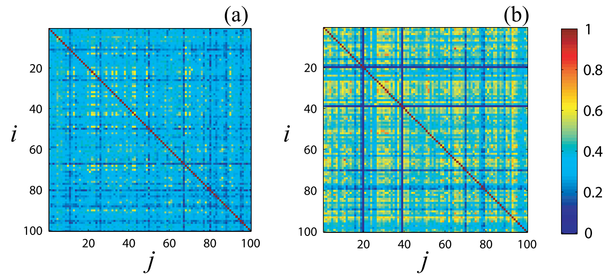

Figure 1: Correlation matrixes of daily return rates $r_i(t)$ in (a)Tokyo Stock Exchange and in (b)New York Stock Exchange. We showed color maps as the mean value from 2009 to 2012.

2.2. Principal-component Portfolio Model

As shown in Figure 1, some stocks have common risk factors and are sometimes synchronized with each other. For this reason, Meucci proposed the principal-component portfolio[2], which uses the principal components of return rates.

First, we diagonalize the covariance matrix $\hat{\boldsymbol{\Sigma}}(t+1)$ estimated by Eq.(4) as follows:

$$\boldsymbol{G}^{\mathrm{t}}\hat{\boldsymbol{\Sigma}}(t+1)\boldsymbol{G} = \mathrm{diag}\left[\lambda_1, \lambda_2, \cdots, \lambda_n\right], \tag{6}$$

where $\boldsymbol{G} = [\boldsymbol{g}_1, \boldsymbol{g}_2, \cdots, \boldsymbol{g}_n]$ and $\{\lambda_i\}$ means respectively eigenvectors and eigenvalues of $\hat{\boldsymbol{\Sigma}}(t+1)$. Then, ith principal-component $r^{\dagger}_i(t)$ can be given by $r^{\dagger}_i(t) = \boldsymbol{g}^{\mathrm{t}}_i\boldsymbol{r}(t)$. The variance of $r^{\dagger}_i(t)$ corresponds to the ith eigenvalue λ_i.

Next, the allocation rates for principal-components $\boldsymbol{d}^{\dagger}(t)$ are given by $\boldsymbol{d}^{\dagger}(t) = \left[\boldsymbol{G}^{-1}\boldsymbol{d}^{\mathrm{t}}(t)\right]^{\mathrm{t}}$. Then, the estimated risk of the original portfolio $\hat{\sigma}^2_p(t+1)$ can be calculated by

$$\hat{\sigma}^2_p(t+1) = \sum_{i=1}^{n} d^{\dagger^2}_i(t)\lambda_i \tag{7}$$

because all principal-components are uncorrelated with each other.

Finally, to optimize $\boldsymbol{d}(t)$, we maximize the variance index N_{Ent}:

$$N_{Ent} = \exp\left(-\sum_{i=1}^{n} p_i \log(p_i)\right), \text{ where } p_i = \frac{d^{\dagger^2}(t)\lambda_i(t)}{\hat{\sigma}^2_p(t+1)}. \tag{8}$$

Here, p_i means the ith risk contribution ratio. Namely, this portfolio model doesn't focus on the return rate $\hat{r}_p(t+1)$ to optimize allocation rates $\boldsymbol{d}(t)$ differently from the Sharpe ratio N_{SR} of Sec. 2.1.

3. Our Proposed Portfolio Model

3.1. Application of a nonlinear prediction model

If a financial market has a slightly dynamics like $r_i(t+1) = \boldsymbol{F}\left[r_i(t), r_i(t-1), \cdots\right]$, there is a possibility to predict its future movements by using a data-mining technique to learning the dynamics $\boldsymbol{F}$ from the past movements. As one of learning techniques, we apply a nonlinear prediction method to estimate future return rates $r_i(t+1)$. Here, let us denote the input of $\boldsymbol{F}$ as $\boldsymbol{v}_i(t) = [r_i(t), r_i(t-\tau_i), \cdots, r_i(t-(k_i-1)\tau_i)]$, where τ_i and k_i correspond to a delay time and an embedding dimension in terms of chaos prediction method. Then, if we consider the dynamics $\boldsymbol{F}$ as a coefficient vector, we can make a prediction model as follows:

$$\hat{r}_i(t+1) = [\boldsymbol{v}_i(t)\ 1] \cdot \hat{\boldsymbol{F}}. \tag{9}$$

If a financial system has a nonlinear dynamics $\boldsymbol{F}$, similar inputs generate similar outputs. Namely, similar past data $\{\boldsymbol{v}_i(t)\}$ are important to learn the dynamics $\boldsymbol{F}$. Therefore, we apply the weighted least square method according to the Euclidian distance $l_i(t, a) = \|\boldsymbol{v}_i(t) - \boldsymbol{v}_i(t-a)\|$, where $\boldsymbol{v}_i(t)$ is the target input and $\boldsymbol{v}_i(t-a)$ is each learning data of the recent L period. Then, the weighted factor $w_i(t, a)$ is given by $w_i(t, a) = \exp(-l_i(t-a))$, and the weighted matrix is composed by $\boldsymbol{W} = \mathrm{diag}[w_i(t, 1), w_i(t, 2), \cdots, w_i(t, L+(k_i-1)\tau_i+1)]$, and the dynamics $\boldsymbol{F}$ can be estimated locally by

$$\hat{\boldsymbol{F}} = \left[\boldsymbol{X}^{\mathrm{t}}\boldsymbol{W}^{\mathrm{t}}\boldsymbol{W}\boldsymbol{X}\right]^{-1}\boldsymbol{X}^{\mathrm{t}}\boldsymbol{W}^{\mathrm{t}}\boldsymbol{W}\boldsymbol{Y}, \tag{10}$$

where

$$\begin{aligned}
\boldsymbol{X} &= \begin{bmatrix} \boldsymbol{v}_i(t-1) & \boldsymbol{v}_i(t-2) & \cdots & \boldsymbol{v}_i(t-L+(k_i-1)\tau_i+1) \\ 1 & 1 & \cdots & 1 \end{bmatrix}^{\mathrm{t}}, \\
\boldsymbol{Y} &= [r_i(t), r_i(t-1), \cdots, r_i(t-L+(k_i-1)\tau_i+2)]^{\mathrm{t}}.
\end{aligned}$$

The estimated coefficients of $\hat{\boldsymbol{F}}$ mean the slope of hyperplane between an input $\boldsymbol{v}_i(t)$ and an output $r_i(t+1)$, but it changes locally according to the state of the input $\boldsymbol{v}_i(t)$. Namely, this estimation corresponds to a nonlinear regression with a hypersurface globally.

Moreover, to confirm the advantage of nonlinear prediction model against the simple moving average in Eq.(3), we perform the Wilcoxon signed-rank test[5], which is a non-parametric statistical hypothesis test. As a result, z score was calculated as $z = 16.705$ by 670 stocks in Tokyo Stock Exchange, and $z = 9.278$ by 499 stocks in New York Stock Exchange. Because these z scores are over 1.645, we can say that the nonlinear prediction works better to predict real financial markets over the 95% significance level.

3.2. Application of the DCC-GARCH Model

The classical portfolio models denote the risk as the variance or the covariance $\sigma_{ij}(t+1)$ in Eq.(4). Here, the risk can be also denoted by the covariance matrix based on the prediction errors $\boldsymbol{e}(t)\,(= \boldsymbol{r}(t) - \hat{\boldsymbol{r}}(t))$ because

$$\begin{aligned}
\boldsymbol{\Sigma}(t) &= \mathrm{Cov}[\boldsymbol{r}(t)] \\
&= \mathrm{E}\left[[\boldsymbol{r}(t)-\hat{\boldsymbol{r}}(t)] \cdot [\boldsymbol{r}(t)-\hat{\boldsymbol{r}}(t)]^{\mathrm{t}}\right] \\
&= \mathrm{E}\left[\boldsymbol{e}(t)\boldsymbol{e}^{\mathrm{t}}(t)\right].
\end{aligned} \tag{11}$$

Moreover, as shown in Figure 2, the nonlinear prediction error between each stock has a long-term autocorrelation structure. This means that the estimation accuracy of the risk can be improved by the DCC model[6], which is the newest multivariate GARCH model and can reduce its numerical cost.

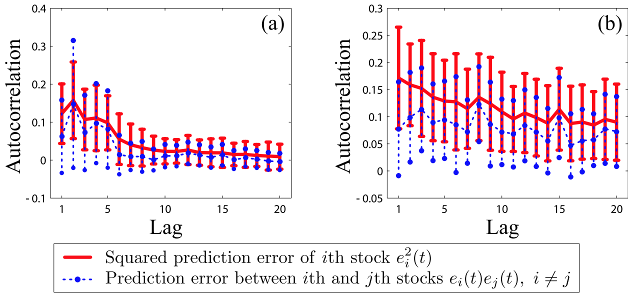

Figure 2: Correlograms of prediction errors $\boldsymbol{e}(t)\boldsymbol{e}^{\mathrm{t}}(t)$ in (a)Tokyo Stock Exchange and in (b)New York Stock Exchange from 2009 to 2012. Each error bar shows the standard deviation.

At the beginning, the DCC model proposed by Engle[6] models the prediction error as follows:

$$\begin{aligned}\boldsymbol{e}(t+1) &\sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}(t+1)) \\ &= \boldsymbol{\Sigma}^{1/2}(t+1)\cdot\boldsymbol{\varepsilon}(t+1),\ \boldsymbol{\varepsilon}(t+1) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{1}), \end{aligned} \quad (12)$$

where, $\boldsymbol{\varepsilon}(t+1)$ means the standard multivariate normal distribution. Then, this model decomposes the conditional covariance matrix based on the nonlinear prediction errors $\boldsymbol{\Sigma}(t+1)$ as follows:

$$\boldsymbol{\Sigma}(t+1) = \boldsymbol{D}(t+1)\boldsymbol{R}(t+1)\boldsymbol{D}(t+1), \quad (13)$$

where $\boldsymbol{D}(t{+}1) = \mathrm{diag}\left[\sigma_1(t+1), \sigma_2(t+1), \cdots, \sigma_n(t+1)\right]$, and $\boldsymbol{R}(t+1)$ means the conditional correlation matrix.

As for the conditional variance $\sigma_i^2(t+1)$ of $\boldsymbol{D}(t+1)$, an univariate GARCH model[7] can be modeled by

$$\begin{aligned}\sigma_i^2(t+1) &= \beta_0 + \beta_1 e_i^2(t) + \beta_2\sigma_i^2(t), \\ &= \beta_0\sum_{a=0}^{\infty}\beta_2^a + \beta_1\sum_{a=0}^{\infty}\beta_2^a e_i^2(t-a). \end{aligned}\quad (14)$$

This is why GARCH model can estimate the conditional variance $\sigma_i^2(t{+}1)$ by the long-term autocorrelation structure of squared nonlinear prediction errors $e_i^2(t)$.

Next, the conditional correlation matrix $\boldsymbol{R}(t+1)$ is modeled by

$$\boldsymbol{R}(t+1) = \boldsymbol{Q}^{*^{-1}}(t+1)\boldsymbol{Q}(t+1)\boldsymbol{Q}^{*^{-1}}(t+1), \quad (15)$$

$$\boldsymbol{Q}(t+1) = (1-\gamma_1-\gamma_2)\bar{\boldsymbol{Q}}(t)+\gamma_1\boldsymbol{\varepsilon}(t)\boldsymbol{\varepsilon}^{\mathrm{t}}(t)+\gamma_2\boldsymbol{Q}(t), \quad (16)$$

$$\bar{\boldsymbol{Q}}(t) = \frac{1}{U}\sum_{a=0}^{U-1}\boldsymbol{\varepsilon}(t-a)\boldsymbol{\varepsilon}^{\mathrm{t}}(t-a), \quad (17)$$

where $\boldsymbol{Q}(t)$ means the conditional covariance matrix of the standardized error $\boldsymbol{\varepsilon}(t)\left(= \left[\boldsymbol{e}^{\mathrm{t}}(t)\boldsymbol{D}^{-1}(t)\right]^{\mathrm{t}}\right)$, and $\bar{\boldsymbol{Q}}(t)$ means the unconditional variance matrix of the standardized error during the last U period. In this study, we set $U = 245$[8]. In addition, we denote $q_{ij}(t+1)$ as an element of $\boldsymbol{Q}(t+1)$, and $\boldsymbol{Q}^*(t+1) = \mathrm{diag}\left[q_{11}^{1/2}(t+1), q_{22}^{1/2}(t+1), \cdots, q_{nn}^{1/2}(t+1)\right]$. Then, Eq.(16) can be rewritten as

$$\begin{aligned}\boldsymbol{Q}(t+1) &= (1-\gamma_1-\gamma_2)\sum_{a=0}^{\infty}\gamma_2^a\bar{\boldsymbol{Q}}(t-a) \\ &\quad + \gamma_1\sum_{a=0}^{\infty}\gamma_2^a\boldsymbol{\varepsilon}(t-a)\boldsymbol{\varepsilon}^{\mathrm{t}}(t-a). \end{aligned}\quad (18)$$

Namely, the DCC model is considered as a multivariate model of the GARCH model. Here, it can be said that the Markowitz's mean-variance portfolio model[1] uses the limited DCC-GARCH model, that is, $\hat{\boldsymbol{r}}(t-a) = \bar{\boldsymbol{r}}(t)\ (0 \le a)$, $\beta_0 = 0, \beta_1 = 1/T, \beta_2 = 1$(if $0 \le a \le T-1$) or $\beta_2 = 0$ (if $T \le a$), $\boldsymbol{\varepsilon}(t-a) = \boldsymbol{r}(t-a) - \bar{\boldsymbol{r}}(t)\ (0 \le a)$, $U = T$, and $\gamma_1 = \gamma_2 = 0$.

Then, we can estimate the model parameters $\beta_a, (a = 1\sim 3)$ and $\gamma_a, (a = 1, 2)$ by the maximum likelihood method, we can estimate the covariance matrix $\hat{\boldsymbol{\Sigma}}(t+1)$ by the DCC model. After that, we substitute it into Eq.(2), substitute the return rate $\hat{r}_i(t+1)$ estimated by the nonlinear prediction in Eq.(9) into Eq.(1), and maximize N_{SR} in Eq.(5) to optimum the allocation rates $\boldsymbol{d}(t)$ of a portfolio. We call it *the nonlinear DCC portfolio model.*

3.3. Application of the Long and Short Strategy

As shown in Fig.1, return rates of each stock have positive correlation. The principal-component portfolio can solve this problem, but for our nonlinear DCC portfolio model the long and short strategy is useful to spread the whole risk of a portfolio. Therefore, we optimize allocation rates $\boldsymbol{d}(t)$ under the condition:

$$\left|\begin{array}{ll}\text{max.} & N_{SR} \\ \text{s.t.} & \sum_{i\in I^+} d_i(t) = 0.5,\ \text{where } I^+ = \{i | d_i(t) > 0\} \\ & \sum_{i\in I^-} d_i(t) = -0.5,\ \text{where } I^- = \{i | d_i(t) < 0\}\end{array}\right. \quad (19)$$

Here, $d_i(t) > 0$ means a long position and $d_i(t) < 0$ means a short position. This strategy can enlarge the portfolio effect as mentioned below. First, Eq.(2) can be rewritten as

$$\hat{\sigma}_p^2(t+1) = \sum_{i=1}^{n} d_i^2(t)\hat{\sigma}_i^2(t+1) + \sum_{i=1}^{n}\sum_{j\neq i}^{n} d_i(t)d_j(t)\hat{\sigma}_{ij}(t+1). \quad (20)$$

Then, as shown in Fig.1, return rates of stocks are sometimes synchronized each other, and therefore $\{\hat{\sigma}_{ij}(t+1)\}$ keep positive values. However, by taking long and short positions simultaneously in Eq.(19), the allocation rates $\boldsymbol{d}(t)$ are mixed by positive and negative values. Therefore, because $d_i(t)d_j(t)\hat{\sigma}_{ij}(t{+}1)$ in Eq.(20) can be less than zero, the whole risk $\hat{\sigma}_p(t+1)$ can be reduced.

4. Investment Simulations

We perform investment simulations with real stock daily data to confirm the validity of the nonlinear DCC portfolio model. The investment period is from 2009 to 2012, and the learning period is from 2001 to 2008 for optimization to the model parameters: T, the embedding dimension k_i, and the delay time τ_i. Additionally, we set the length of learning data L to the half length of the learning period, that is, only four years to apply the cross-validation method[8] to avoid overfitting. Moreover, to compose a multi-stock portfolio with n stocks, we preferentially selected more predictable stocks having higher likelihood of each prediction model for the learning data, following the previous study[8].

Figure 3 shows the investment performance. Firstly, the asset growth rate η is calculated by $\eta = M(\mathrm{end})/M(1)$,

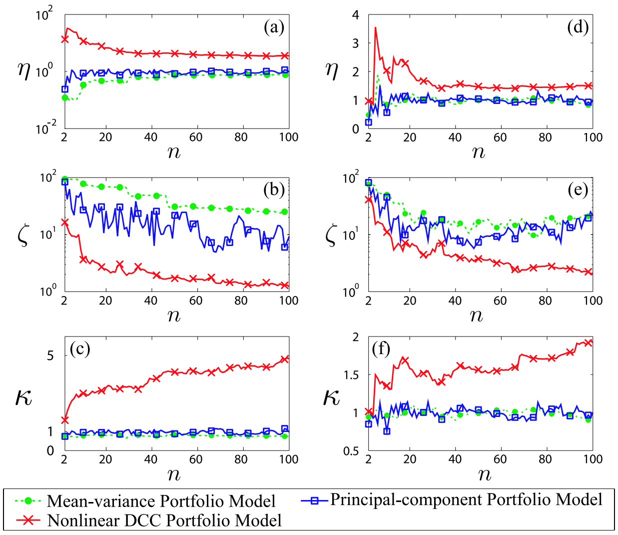

Figure 3: Investment performance of each portfolio model in (a)–(c)Tokyo Stock Exchange and in (d)–(f)New York Stock Exchange.

where $M(t)$ means the present asset at the time of t, $M(1)$ means the initial asset, and $M(\text{end})$ means the final asset. If $\eta > 1$, we can get profits, and larger η is better. Secondly, the maximum draw-down rate ζ [%] is calculated by

$$\zeta = \max_{1 \le t' \le t}\{r_d(t')\}, \text{ where } r_d(t) = \left(1 - \frac{M(t)}{\max_{1 \le t' \le t}\{M(t')\}}\right) \times 100. \quad (21)$$

This means how much we lost our asset so far, that is, the degree of possible danger of an investment strategy. Namely, smaller $r_d(t)$ is better and safer. Thirdly, the profit factor κ [times] is calculated by

$$\kappa = \frac{\sum_{t'=1}^{t}\{\Delta M(t') | \Delta M(t') \ge 0\}}{\sum_{t'=1}^{t}\{\Delta M(t') | \Delta M(t') < 0\}}, \qquad (22)$$

where $\Delta M(t) = M(t) - M(t-1) = (1 + r_p(t))M(t-1)$. This shows the efficiency of investment. These are commonly used to evaluate trading performance.

As shown in Figs.3(a) and (d), the mean-variance portfolio model cannot obtain enough profits, and shows a huge draw-down rate in Figs.3(b) and (e). In addition, in Figs.3(c) and (f), the profit factor κ are almost 1[times]. Therefore, this portfolio model is theoretical but not practical. One of the reason is that the power of prediction like the simple moving average of Eq.(3) is weak to predict real markets[8].

Secondly, the principal-component portfolio model also cannot obtain profits in Figs.3(a) and (d). However, as shown in Figs.3(b) and (e), it can reduce draw-down rates more than the mean-variance portfolio model, but still Figs.3(c) and (f) show that the profit factor κ are almost 1[times].

Finally, as for our the nonlinear DCC portfolio model, we can confirm that the asset growth rate η and the profit factor κ are improved greatly. Moreover, the maximum draw-down rate is also reduced obviously. Therefore, we can say that this portfolio model can manage our asset more efficient and safer. Then, if the number of stocks n to compose a portfolio increases, the asset growth rate η becomes smaller, but the maximum draw-down rate ζ can be more reduced. Namely, we can select the optimal number of stocks according to the risk tolerance of each investor.

5. Conclusion

Markowitz's mean-variance portfolio model has two problems. The first one is that it is difficult to reduce the total risk of a portfolio because some stocks synchronized with each other. For this reason, Meucci proposed the principal-component portfolio model, which makes uncorrelated composite stocks by the principal component analysis and compose a portfolio based on these composite stocks. However, even this portfolio model cannot solve the second problem of the mean-variance portfolio model because the estimation technique of a future expected return rate and a risk are so simple that their prediction power is weak. Therefore, the principal-component portfolio model can reduce draw-down than the mean-variance portfolio model, but cannot increase the asset growth rate.

In the present study, we tried to solve both problems. To solve the second problem, we used the nonlinear prediction model to improve the estimation accuracy of future return rates, and used the DCC-GARCH model to estimate risks based on prediction errors, which have long-term memory. Moreover, to solve the first problem, we applied the long and short strategy to our nonlinear DCC portfolio model for enhancing the portfolio effect. Finally, through investment simulations with real stock data, we confirmed that our portfolio strategy can work well not only to enlarge profits but also to reduce risks of the portfolio.

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