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Complex Particle Swarm Optimization

Kenya Jin'no, Takuya Shindo

Electrical & Electronics Engineering,
Nippon Institute of Technology
Miyashiro, Minami-Saitama, Saitama, 34-8502, Japan
Email: jinno@nit.ac.jp

Abstract—Particle Swarm Optimization (PSO) system is one of the powerful algorithm for solving global optimization problems. A lot of research results about the searching ability of PSO system have been reported, however, there are few ones which paid attention to the dynamics of PSO system. In order to analyze the dynamics of PSO system, M. Clerc and J. Kennedy have derived a generalized model of the PSO system. And they have analyzed the stability of the system based on the eigenvalue of the system. In this article, we propose a novel canonical deterministic PSO system, and analyze the relation between the eigenvalues and the searching ability. Moreover, we propose a reacceleration PSO system. We will clarify the searching ability of the proposed reacceleration PSO system.

1. Introduction

Searching for an optimal value of given evaluation function to various problems is very important in the engineering field. In order to solve such optimization problems speedy, various kinds of heuristic optimization algorithms have been proposed. Particle Swarm Optimization (PSO), which was originally proposed by J.Kennedy *et al.*[1][2], is one one of such heuristic algorithms.

The original PSO is described as following equations:

$$\boldsymbol{v}_j^{t+1} = w\boldsymbol{v}_j^t + c_1 r_1(\boldsymbol{pbest}_j^t - \boldsymbol{x}_j^t) + c_2 r_2(\boldsymbol{gbest}^t - \boldsymbol{x}_j^t) \quad (1)$$

$$\boldsymbol{x}_j^{t+1} = \boldsymbol{x}_k^t + \boldsymbol{v}_j^{t+1}. \quad (2)$$

where, $w \geq 0$ is an inertia weight coefficient, $c_1 \geq 0$, and $c_2 \geq 0$ are acceleration coefficients, $|r_1| \leq 0$, and $|r_2| \leq 0$ are two separately generated uniformly distributed random numbers in the range $[0, 1]$. $\boldsymbol{x}_j^t \in \Re^n$ denotes the location of the j-th particle on the t-th iteration in the n dimensional space, and $\boldsymbol{v} \in \Re^n$ denotes the velocity of the j-th particle on the t-th iteration. $\boldsymbol{pbest}_j^t \in \Re^n$ means the best location which gives the best value of the evaluation function, of the j-th particle on the t-th iteration , and $\boldsymbol{gbest}^t \in \Re^n$ means the global best location which gives the best value of the evaluation function on the t-th iteration in the swarm.

Equations (1) and (2) are described the dynamics of the j-th particle. Each particle shares an information of a current optimal value of the evaluation function, and the corresponding location of the best particle. Also, each particle memorizes own best record of the evaluation value and its location. The moving direction is calculated by using such information. Namely, all particles will move toward to a coordinate which gives current best value of the evaluation function.

Since the acceleration coefficients are multiplied by a random number, the system can be regarded as a stochastic system. Therefore, the rigorous analysis of such system is difficult. In order to analyze the dynamics of such system, M. Clerc and J. Kennedy proposed the following simple formula[3] with the simplicity acceleration coefficients as

$$\boldsymbol{p}_j = \frac{c_1 \boldsymbol{pbest}_j + c_2 \boldsymbol{gbest}}{c_1 + c_2} \quad (3)$$

Also, they eliminate the random number from the system, therefore, the system is said to be deterministic system. In this case, the system can be described as

$$\begin{cases} \boldsymbol{v}_j^{t+1} = w\boldsymbol{v}_j^t + c(\boldsymbol{p}_j^t - \boldsymbol{x}_j^t), \\ \boldsymbol{x}_j^{t+1} = \boldsymbol{x}_k^t + \boldsymbol{v}_j^{t+1}, \end{cases} \quad (4)$$

where, $c = c_1 + c_2$ denotes an acceleration coefficient.

The system of Eqs.(4) can be transformed as the following matrix form.

$$\begin{bmatrix} \boldsymbol{v}_j^{t+1} \\ \boldsymbol{y}_j^{t+1} \end{bmatrix} = \begin{bmatrix} w & -c \\ w & 1-c \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_j^t \\ \boldsymbol{y}_j^t \end{bmatrix} \quad (5)$$

where, $\boldsymbol{y}_j^t = \boldsymbol{p} - \boldsymbol{x}^t$.

Note that each dimension component in $\boldsymbol{v}$ and $\boldsymbol{y}$ is independent, therefore the behavior of the system is characterized by the eigenvalues of the matrix in Eq. (5)

The eigenvalue λ of the matrix is given as

$$\lambda = \frac{(w+1-c) \pm \sqrt{c^2 - 2c(w+1) + (w-1)^2}}{2} \quad (6)$$

If $c^2 - 2c(w+1) + (w-1)^2 < 0$ is satisfied, the eigenvalue is complex number. Namely, if

$$w + 1 - 2\sqrt{w} < c < w + 1 + 2\sqrt{w} \quad (7)$$

is satisfied, the eigenvalue of the system (5) is complex number. Figure 1 illustrates the region of (7). If the parameters is an upper value on the curve in Fig. 1, the eigenvalue becomes a complex number.

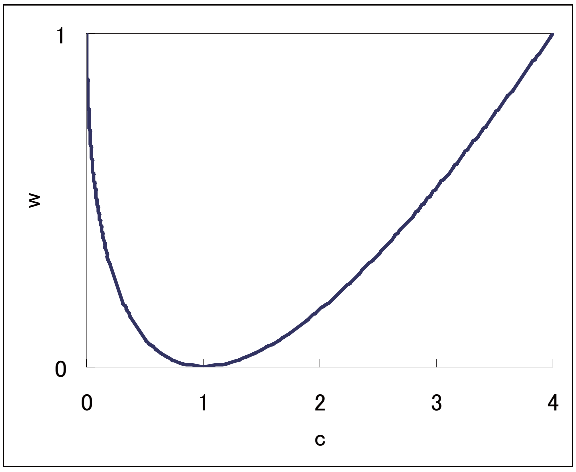

Figure 1: The parameters region for complex eigenvalue.

The system of (5) is a discrete-time system. For the system to become stable, the eigenvalues must exist within the unit circle on the complex plane. Therefore, we can derive the following theorem.

[Theorem] The system is said to be stable iff the following condition is satisfied.

$$\begin{array}{l} 0 < w < 1 \text{ for } w + 1 - 2\sqrt{w} < c < w + 1 + 2\sqrt{w} \\ |w + 1 - c| < 2 - \sqrt{c^2 - c(w+1) + (w-1)^2} \\ \qquad \text{for } c \le w + 1 - 2\sqrt{w} \text{ or } w + 1 + 2\sqrt{w} \le c \end{array} \tag{8}$$

If the parameters satisfy above condition, the system must converge to a fixed point.

2. PSO with complex eigenvalue

In our previous work[9], we confirmed that the PSO system exhibits remarkable searching ability in the case where the eigenvalues are complex number. Therefore, in this article, we will pay attention to the case where the eigenvalue is complex number. In this case, Eqs.(5) can be transformed into the following canonical matrix form as

$$\begin{bmatrix} v_j^{t+1} \\ y_j^{t+1} \end{bmatrix} = \begin{bmatrix} \delta & -\omega \\ \omega & \delta \end{bmatrix} \begin{bmatrix} v_j^t \\ y_j^t \end{bmatrix} \tag{9}$$

The eigenvalues of Eq.(9) are complex conjugate number as

$$\lambda = \delta \pm j\omega. \tag{10}$$

In this case, the system of (9) can be described as the following form, too.

$$\begin{bmatrix} v_j^{t+1} \\ y_j^{t+1} \end{bmatrix} = \sqrt{\delta^2 + \omega^2} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_j^t \\ y_j^t \end{bmatrix} \tag{11}$$

where, $\theta = \arctan\frac{\omega}{\delta}$.

Namely, the trajectory of the system (11) must converge to the origin with spiral motion. The rotation angle is $\theta = \arctan\frac{\omega}{\delta}$, and the damping factor is given as $\sqrt{\delta^2 + \omega^2}$.

Figure 2 illustrates an example of the trajectory in the phase space on $\delta = 0.70$, and $\omega 0.70$. In this case, the tra-

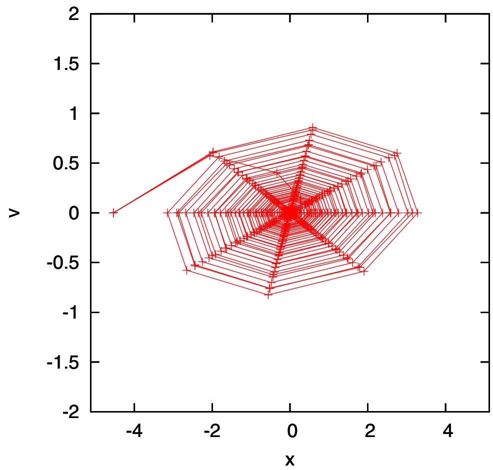

Figure 2: The trajectory in the phase space x_{11} vs v_{11} ($N = 2, \delta = 0.70, \omega = 0.70$)

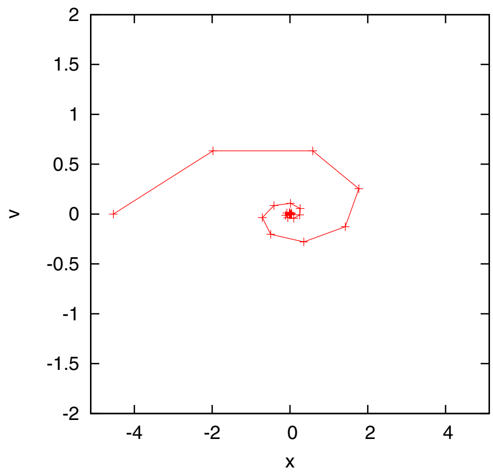

Figure 3: The trajectory in the phase space x_{11} vs v_{11} ($N = 2, \delta = 0.60, \omega = 0.60$)

jectory rotates $\pi/4$ [rad] around the origin on each iteration, and the coordinate is attenuated for the origin.

Figure 3 illustrates another example of the trajectory which can be observed in the case of $delta = 0.60$, and $\omega = 0.60$. As for this case, too, the trajectory rotates $\pi/4$ [rad] around the origin on each iteration. However, the attenuation effect is different, the trajectory converges to the origin more quickly than the previous case. When the attenuation coefficient $\sqrt{\delta^2 + \omega^2}$ is close proximity to 1.0,

Table 1: Benchmark problems for the numerical simulations.

function name	formula	minimum value
Sphere	$F_{Sphere}(x_d) = \sum_{d=1}^{D} x_d^2$	$F_{Sphere}(0,0,0,\dots,0) = 0$
Rosenbrock	$F_{Rosenbrock}(x_d) = \sum_{d=1}^{D-1}(100(x_{d+1} - x_d^2)^2 + (x_d - 1)^2)$	$F_{Rosenbrock}(1,1,1,\dots,1) = 0$
Rastrigin	$F_{Rastrigin}(x_d) = 10D + \sum_{d=1}^{D}((x_d)^2 - 10\cos(2\pi x_d))$	$F_{Rastrigin}(0,0,0,\dots,0) = 0$
Griewank	$F_{Griewank}(x_d) = 1 + \frac{1}{4000}\sum_{d=1}^{D} x_d^2 - \prod_{d=1}^{D}\cos\left(\frac{x_d}{\sqrt{d}}\right)$	$F_{Griewank}(0,0,0,\dots,0) = 0$
Schaffer's f6	$F_{Schaffer}(x_d) = 0.5 - \frac{\left(\sin\sqrt{x_d^2 + x_{d+1}^2}\right)^2 - 0.5}{(1.0 + 0.001(x_d^2 + x_{d+1}^2))^2}$	$F_{Schaffer}(0,0,0,\dots,0) = 1$

the system can search wide region, but the system needs a long time to converge. In order to confirm this fact, we carry out some numerical simulations by using the following well-known benchmark problems as shown in Table 1. For such benchmark problems, we confirm the performance of the deterministic PSO depending on the parameters. Each objective functions are consisted with 10 dimensions variables. 10 particles are contained in the swarm of our proposed PSO. We observed about finding optimum value which is gotten after the constant elapse time. To ascertain the influence of the eigenvalues, for each eigenvalues, we carry out 10 numerical simulation which changes its initial values. The results adopt their average of the error from the optimal value.

Figure 4 shows the results. Each point shows the parameter which gives a searching result as much as higher rank 10 to each objective function. This result indicates that the optimal value of the eigenvalue exists without depending on the objective function is suggested.

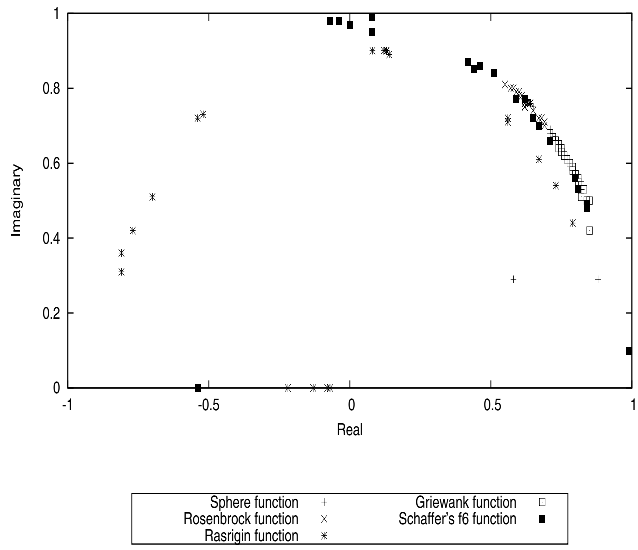

Figure 4: The parameters that good searching results are given. "Real" means δ, and "Imaginary" means ω.

3. Reacceleraion PSO

Based on the result of previous section, we can say that the large damping factor is important to the ability to search for the deterministic PSO system. As shown in Figs. 2 and 3, in this case, the convergence time becomes long. When setting parameters to converge quickly, the system does not search sufficiently. To overcome this problem, we propose a "Reacceleration PSO". If a component of the velocity vector $\boldsymbol{v}_j$ converges to zero, a random perturbation is added to the component. As a result of this operation, the system searches again.

Figures 5 and 6 shows the example of the time fluctuation of the velocity. In the case of Fig. 5, the system does not search when the velocity converges once. On the other hand, in the case of Fig. 6, the system can continue a search. By combining this feature and parameters which correspond to the small damping factor, the system becomes able to search effectively. Figure 7 illustrates the comparison of the fluctuation of the searched optimal value between the deterministic PSO system and the reacceleration PSO system. In case of the deterministic PSO system, after the system converges around a local minimum, because it does not search global area, the optimal solution could not be searched. On the other hand, the reacceleration PSO system can search the optimal solution because it begins searching again after it converges once.

4. Conclusions

In this article, we have analyze a deterministic PSO system. We have clarified the relation between the eigenvalues and the ability to search. Namely, the system has the large damping factor exhibits the effective searching ability. However, the large damping factor causes the slow convergence. In order to converge speedy, we have proposed a reacceleraion PSO system. We have confirmed that the reacceleraion PSO system has excellent ability to search optimal solution.

References

[1] James Kennedy and Rusell Eberhart, "Particle Swarm Optimization", in *Proc. IEEE Int. Conf. Neural Networks,* pp. 1942-1948, 1995.

[2] James Kennedy, "The particle swarm: Social adaptation of knowledge," in *Proc. IEEE Int. Conf. Evolutionary Computation,* pp. 303-308, 1997.

[3] Maurice Clerc, and James Kennedy, "The Particle Swarm – Explosion, Stability, and Convergence in a Multidimensional Complex Space," *IEEE Trans. Evol. Comput.* , vol. 6, no. 1, pp. 58-73, 2002.

[4] Visakan Kaidirkamanathan, Kirusnapillai Selvarajah, and Peter J. Fleming, "Stability Analysis of the Particle Dynamics in Particle Swarm Optimizer," *IEEE Trans. Evol. Comput.* , vol. 10, no. 3, pp. 245-255, 2006.

[5] Asanga Ratnaweera, Saman K. Halgamuge, and Harry C. Watson, "Self-Organizing Hierarchical Particle Swarm Optimizer With Time-Varying Acceleration Coefficients," *IEEE Trans. Evol. Comput.* , vol. 8, no. 3, pp. 240-255, 2004.

[6] Rui Mendes, James Kennedy, and José Noves, "The Fully Informed Particle Swarm: Simpler, Maybe Better," *IEEE Trans. Evol. Comput.* , vol. 8, no. 3, pp. 204-210, 2004.

[7] Frans van den Bergh, and Andries P. Engelbrecht, "A Cooperative Approach to Particle Swarm Optimization," *IEEE Trans. Evol. Comput.* , vol. 8, no. 3, pp. 225-239, 2004.

[8] Kenya Jin'no, and Kazuyuki Aihara, "On a searching ability of DT-PSO," *IEICE Technical Report*, NLP2008-133, 2009.

[9] Kenya Jin'no, and Kazuyuki Aihara, "On A Second-Order Hysteresis Neural Network," *2009 RISP Int'l Workshop on Nonlinear Circuits and Signal Processing*, pp. 360-363, 2009.

[10] Takuya Shindo, and Kenya Jin'no, "The Influence of Control Parameters on Searching Ability of PSO Systems," *IEICE Technical Report*, NLP2008-, 2009.

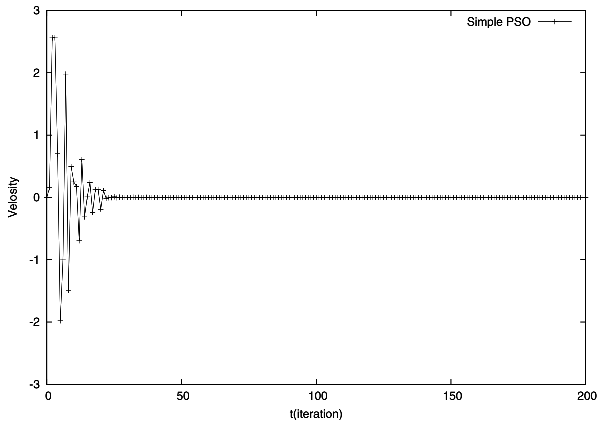

Figure 5: The example of the time fluctuation of the velocity of a deterministic particle. ($\delta = 0.5$, $\omega = 0.5$)

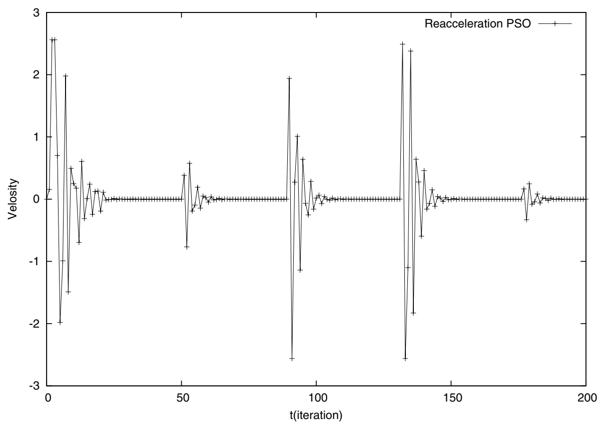

Figure 6: The example of the time fluctuation of the velocity of a particle with reacceleration operation. ($\delta = 0.5$, $\omega = 0.5$)

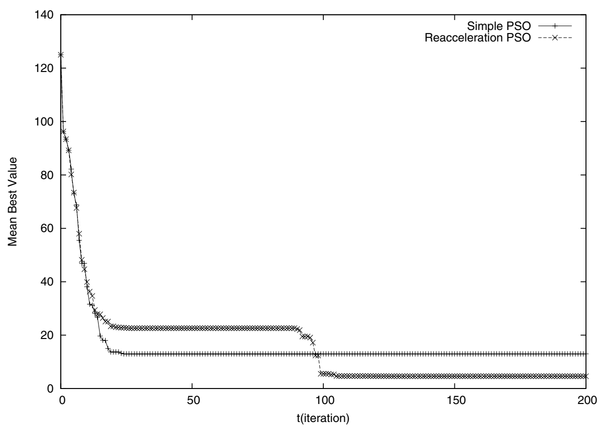

Figure 7: The comparison of the fluctuation of the searched optimal value between the deterministic PSO and the reacceleration PSO. ($\delta = -0.5$, $\omega = 0.5$)