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Construction of Scale-Free Networks with Adjustable Clustering

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Abstract— A complex network is characterized by its degree distribution and clustering coefficient. Given a scale-free network, we propose a node-reconnection algorithm that can alter the clustering coefficient of the network while keeping the degree of each node unchanged. Results are shown when the algorithm is applied to reconnect the nodes of scale-free networks constructed using the Barabási-Albert (BA) model.

1. Introduction

The topology of a network can typically be characterized by three main parameters, namely clustering coefficient, path length and degree distribution. A *regular lattice* exhibits high clustering and large path length whereas a *random network* possesses low clustering and short path length [1]. Both networks fail to characterize many real networks, such as social network, movie-actor-collaboration network and citation network [2, 3], which have large clustering, short path length and power-law degree distribution.

In [2], Watts and Strogatz introduced the *small-world network*, which is characterized by large clustering and short path length. One way to construct a small-world network is by reconnecting the links of a regular network [2]. By tuning a control parameter during the reconnection process, the clustering and path-length properties of the resulting small-world network can be varied. When an extreme value of the control parameter is selected, moreover, a random network is formed. Hence, the small-world network is also regarded as a transition from the regular network to the random network. The small-world network constructed as such, however, has a similar degree distribution of a random network, failing to mimic the power-law degree distribution of many real networks.

Consider a network with power-law degree distribution [4]. (Such a network is also termed as *scale-free network*.) It has been found that when a new node is added to such a network, the new connections associated with this new node are made according to a preferential rule rather than in a random manner. Precisely, the new connections are made with a higher chance to nodes with higher degrees than those with lower degrees. Based on the preferential rule, Barabási & Albert [3] proposed a model (BA model) for the construction of a scale-free network. Several modified models emerged since then [3, 5, 6]. However, the clustering property has not been considered in the construction of the BA network. For a large network, the clustering coefficient is close to zero.

While a small-world network has a high clustering and a scale-free network possesses a power-law degree distribution given by $P(k) \propto k^{-\gamma}$, where $P(k)$ is the probability of a node with degree k and γ is the scaling exponent, the two types of networks may exist independently of each other. Since many real networks possess both small-world and scale-free properties, a number of techniques have been proposed to construct such networks [7]–[14]. In [7], a simple clustered scale-free network with $\gamma = 3$ has been introduced. Other methods to vary the clustering of a scale-free network with a scaling exponent of 3 have also been studied [8, 9, 10]. In [11], small-world, scale-free networks with a wide range of γ are constructed, while the clustering of the networks varies accordingly with the value of γ.

In this paper, we propose a construction method for small-world, scale-free networks with any given scaling exponent and clustering. First, we make use of the BA or modified BA methods to construct a scale-free network with the given scaling exponent. Then, we apply an algorithm to reconnect the links between the nodes while keeping the degrees of the nodes unchanged. Through the reconnection process, the clustering of the scale-free network can be adjusted.

2. Network Construction

2.1. Construction of Scale-Free Networks

First, we generate a scale-free network based on the BA or modified BA models [5, 6]. Denoting the degree of the ith node of the network by k_i, the evolving algorithm and property of the modified BA model can be summarized as follows.

1. *Initialization:* Start with a network with $m_0 = m$ isolated nodes.
2. *Network growth:* At every time step, a new node is added.
3. *Preferential attachment:* m outgoing links are to be added to the new node. Given the parameter $m + k_0$, which is a positive constant representing the initial attractiveness. The probability for the new node to connect to Node i is proportional to k_i+k_0 $(-m < k_0 < \infty)$.

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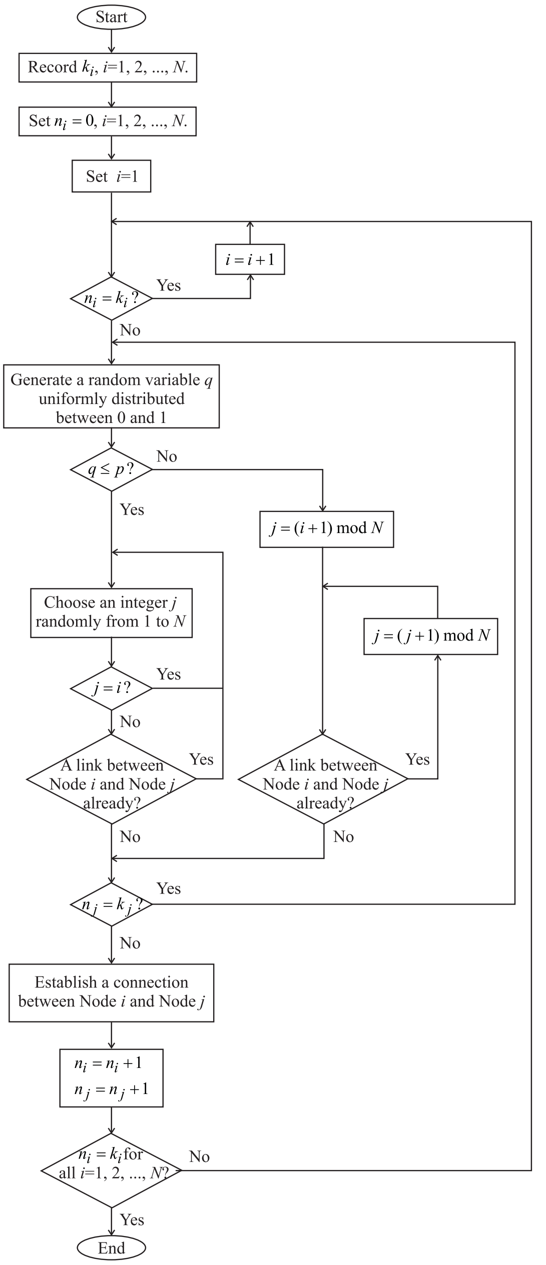

Figure 1: Flow chart of the reconnection process.

4. *Scaling exponent:* The exponent for the modified BA model is computed from $\gamma = 3 + k_0/m$. When $k_0 = 0$, the model degenerates to the BA model and $\gamma = 3$.

2.2. Reconnection of Links

A network in which the degree of nodes follows a power-law distribution has now been formed. Suppose a total of N nodes have been created. We label the nodes by $1, 2, \ldots, N$, according to the order of generation. After recording the original degree of each of the nodes (represented by $k_i, i = 1, 2, \ldots, N$), we remove all links in the network.

We select a value between 0 and 1 for the control parameter p, and reset the number of established connections of all nodes, denoted by $n_i, i = 1, 2, \ldots, N$, to zero. We then reconnect the links of the nodes sequentially according to their node numbers. As we begin with the first node, we set $i = 1$. The reconnection algorithm is described as follows.

1. *Check whether all k_i links of Node i are established:* Compare the number of established connections n_i with the original degree of the node k_i. If $n_i = k_i$, set $i = i + 1$ and repeat Step 1.

2. *Find a Node j which is not connected to Node i:* Generate a random variable q uniformly distributed between 0 and 1.

   (a) If $q \le p$, choose an integer j randomly from 1 to N with equal probability. If $j = i$ or there is already a link between Node i and Node j, repeat Step 2(a).

   (b) If $q > p$, set $j = (i + 1) \bmod N$.

   (c) If there is already a link between Node i and Node j, set $j = (j + 1) \bmod N$ and repeat Step 2(c).

3. *Check whether all k_j links of Node j are established:* If $n_j = k_j$, go to Step 2.

4. *Link establishment:* Establish a connection between Node i and Node j. Increment the values of n_i and n_j by 1.

5. *Check whether the reconnection process of all N nodes has been completed:* If $n_i = k_i, i = 1, 2, \ldots, N$, exit the reconnection algorithm; otherwise, go to Step 1.

The flow diagram of the reconnection process is also depicted in Fig. 1.

Figure 2 shows an example of the structures of the resultant scale-free networks after the reconnection process. Note that for the same node, it has the same degree in the two networks shown. In this example, the degrees of the nodes are first obtained by generating a BA model with $\gamma = 3$. Moreover, there are altogether 30 nodes and $m = 3$ outgoing links are added to each new node during the construction of the scale-free network.

When the control parameter p is set to 0, based on Step 2(b) in the aforementioned algorithm, it can be visualized that more links connecting nodes in the neighborhood are established. Hence, a scale-free network with a high clustering is formed. In Fig. 2(a), it can be observed that the nodes are organized into 4 clusters. Cluster 1 is the largest while Cluster 4 is the smallest. The nodes with larger degrees are mainly found in the largest cluster. In Cluster 4, the nodes possess smaller number of degrees. Moreover, only one common node exists between two clusters. For example, both Cluster 1 and Cluster 2 contain the common

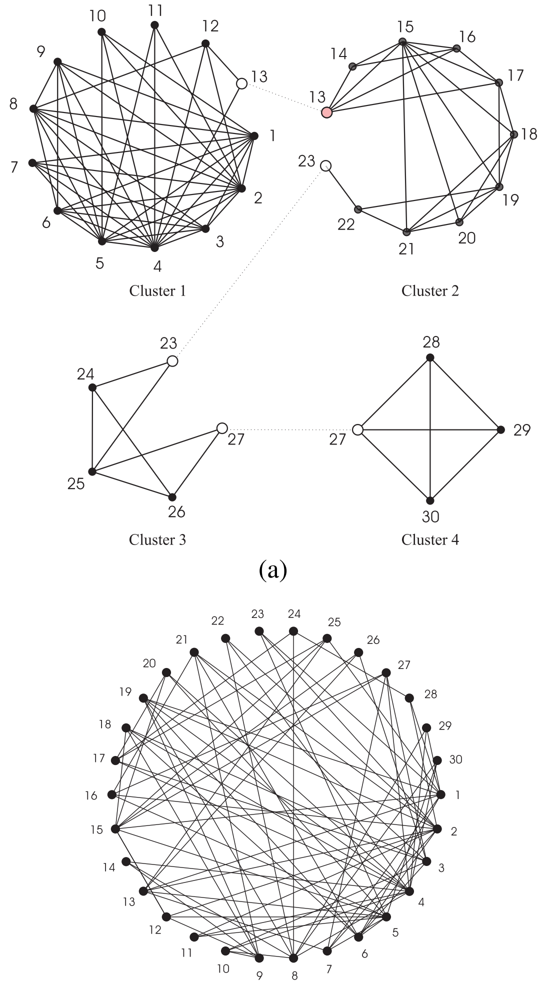

Figure 2: Scale-free networks after links reconnection. (a) p set to 0 and a scale-free network with large clustering is formed. (b) p set to 1 and a scale-free random network is formed.

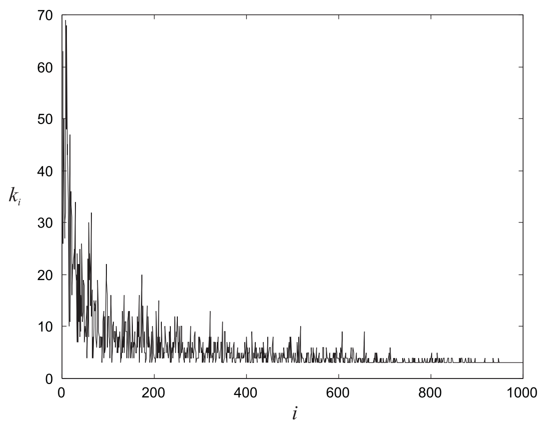

Figure 3: Plot of degree k_i versus node number i of a typical scale-free network generated by the BA model.

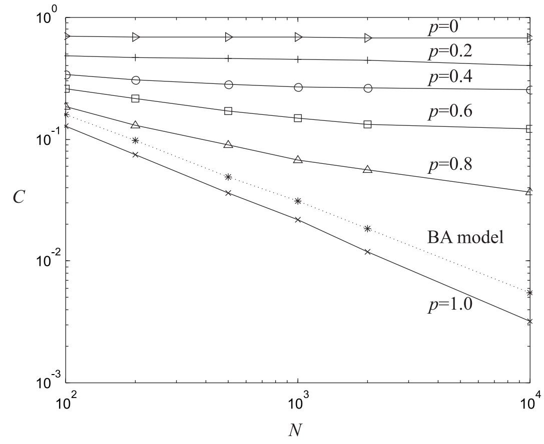

Figure 4: Plot of clustering coefficient C versus number of nodes N for the original (BA) and reconnected scale-free networks.

Node 13, whereas Cluster 2 and Cluster 3 have a common Node 23.

When the value of p increases, the number of links connecting a node with its neighbors becomes smaller and more links are added between nodes which are further apart. As a result, the clustering coefficient is reduced. When $p = 1$, all links are added randomly and the node connects to its neighbors occasionally. Hence, the clustering coefficient is minimized. Such a scale-free random network is shown in Fig. 2(b).

3. Results and Discussions

In the following, we focus our discussions on scale-free networks constructed based on the BA model with $m = 3$. Fig. 3 plots a typical graph showing the degree k_i versus the node number i. Altogether, there are 10^3 nodes ($N = 10^3$). It can be found that in general, nodes with smaller node numbers have higher degrees.

Given a particular value of p. We reconnect the nodes and investigate the variation of the clustering coefficient with p. Note that the degree of each node remains the same before and after the reconnection process. Fig. 4 plots the clustering coefficient, denoted by C, versus N for $p = 0, 0.2, 0.4, 0.6, 0.8$ and 1.0, respectively. It is observed that the clustering coefficient decreases with the control parameter p. For example, when $N = 10^4$, C equals 0.676, 0.123 and 0.0032 for p equals 0, 0.6 and 1, respectively. Moreover, the clustering coefficient decreases in general as the number of nodes N increases from 10^2 to 10^4. But the clustering coefficient also tends to converge to a constant value. For instance, when $p = 0.4$, the clustering coefficient maintains a value of around 0.26 even though the number of nodes increases from 2×10^3 to 10^4. The clustering coefficient of the original scale-free network constructed using the BA model is also plotted on the same graph (dotted line). It can be seen that the model corre-

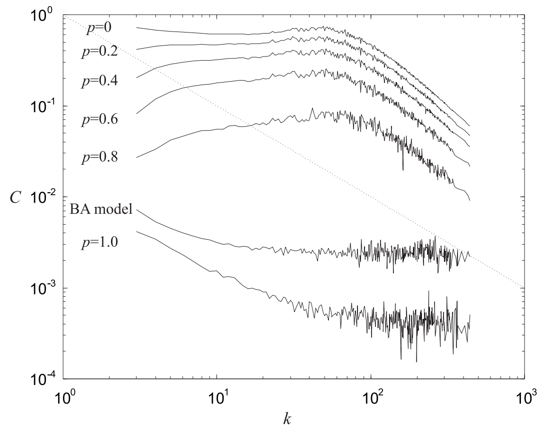

Figure 5: Plot of clustering coefficient C versus node degree k for the original (BA) and reconnected scale-free networks.

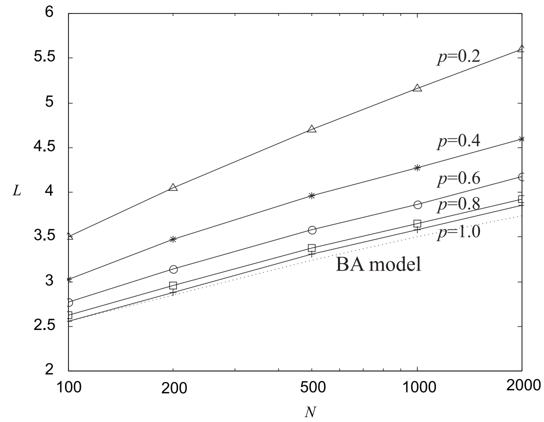

Figure 6: Plot of path length L versus number of nodes N for the original (BA) and reconnected scale-free networks.

sponds to a value of p between 0.8 and 1.0 in our reconnection algorithm.

In Fig. 5, the statistical average of the clustering coefficient C is plotted versus the node degree k for various values of p. We also plot $C = k^{-1}$, as shown by the dotted line, for comparison purpose. It is found that for $p = 0, 0.2, 0.4, 0.6$ or 0.8 and $k > 50$, the value of the clustering coefficient follows a power-law distribution, i.e., $C \sim k^{-1}$. Such an observation is consistent with those reported in [15, 11], where networks have a hierarchical structure. Moreover, it can be seen that for a given value of node degree k, the clustering coefficient C decreases with p. In addition, for the BA model or when $p = 1$, the clustering coefficient becomes independent of the node degree k.

Finally, Fig. 6 depicts the path length L versus the number of nodes N for different values of p. The dotted line shows the results for the BA model. In general, the path length decreases with p but increases logarithmically with the number of nodes N. It is also observed that the path length for the BA model is slightly lower than the case corresponding to $p = 1.0$.

4. Conclusions

A simple method has been proposed to construct small-world, scale-free networks. The main contribution of this work is an algorithm reconnecting the links of a scale-free network to arrive at different clustering coefficients. Based on the algorithm, the degree of each node remains the same before and after the reconnection process. Scale-free networks with a scaling exponent of 3 have been studied and it has been shown that the clustering property can be adjusted successfully based on the proposed method.

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