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A note on the white noise limit of dynamical systems driven by colored noise

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Abstract—The derivation of the reduced equation in the white noise limit for the system driven by colored noise is considered. The reduced equation is directly derived without using the Fokker-Planck equation for the probability distribution which is used in the earlier study.

1. Introduction

The reduction of the dynamical systems to low-dimensional systems, such as the center manifold reduction of the bifurcation theory[1] and the phase dynamics by Kuramoto[2], is a very powerful tool for the theoretical analysis of nonlinear phenomena. Recently, the role of external noise on the reduced phase dynamics has been investigated[3, 4, 5, 6, 7]. The problem considered in the present work is similar to that for the phase reduction of the limit cycle oscillator[5, 6], where the systems driven by the Ornstein-Uhlenbeck process are considered and the reduction is based on the Fokker-Planck equation for the probability equation not for the path itself. In the present study, a simple direct derivation is proposed.

We consider the dynamical system driven by colored noise:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) + \boldsymbol{h}(\boldsymbol{x},\boldsymbol{y})z_\epsilon \tag{1}$$

$$\frac{d\boldsymbol{y}}{dt} = -\frac{1}{\epsilon}\gamma(\boldsymbol{x})\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})z_\epsilon. \tag{2}$$

Here $\epsilon > 0$ is a small parameter and z_ϵ denotes the one-dimensional colored noise satisfying

$$\langle z_\epsilon(t)z_\epsilon(s)\rangle = \frac{1}{\epsilon}\mathrm{e}^{-\frac{|t-s|}{\epsilon}}, \tag{3}$$

where $\langle\cdot\rangle$ denotes the statistical average. Moreover, with the ϵ-independent noise z,

$$z_\epsilon(t) \equiv \frac{1}{\sqrt{\epsilon}} z\left(\frac{t}{\epsilon}\right) \tag{4}$$

is assumed. The matrix-valued function $\gamma(\boldsymbol{x})$ is assumed to have no eigenvalues with zero or negative real parts everywhere and $\boldsymbol{y}$ is assumed to vanish for large limit of t in the absence of noise. The variable $\boldsymbol{x}$ describes the reduced system. γ gives the ratio between the correlation decay rate ϵ of the noise and the decay rates of $\boldsymbol{y}$.

The problem is to derive the closed system of equations for $\boldsymbol{x}$ in the small limit of ϵ, i.e., the white noise limit. In the textbooks on noisy phenomena, similar but different problems are formulated as "white noise process as a limit of nonwhite process" and "adiabatic elimination of first variables". The problem investigated here is considered to be a kind of coupled version of the two problems[8, 9, 10].

In the next section, the result of the white noise limit is presented. A short summary is given in the last section.

2. White noise limit

For small $s - t > 0$, Eq. (2) leads to

$$\boldsymbol{y}(s) - \boldsymbol{y}(t) \simeq \int_t^s \mathrm{e}^{-\frac{1}{\epsilon}\gamma(\boldsymbol{x}(t))(s-t')} z_\epsilon(t')dt'\,\boldsymbol{g}(\boldsymbol{x}(t),\boldsymbol{y}(t)), \tag{5}$$

were the terms larger than $\mathrm{O}(s-t)$ are kept. Up to terms of $\mathrm{O}(s-t)$, we have

$$\boldsymbol{x}(t+\Delta t)-\boldsymbol{x}(t) \simeq \boldsymbol{f}(\boldsymbol{x}(t),\boldsymbol{y}(t))\Delta t+\int_t^{t+\Delta t}\boldsymbol{h}(\boldsymbol{x}(s),\boldsymbol{y}(s))z_\epsilon(s)ds. \tag{6}$$

By inserting Eq. (5), the last term of Eq. (6) is approximated as

$$\begin{aligned}&\int_t^{t+\Delta t}\boldsymbol{h}(\boldsymbol{x}(s),\boldsymbol{y}(t))z_\epsilon(s)ds \\ &\quad+\nabla_y\boldsymbol{h}(\boldsymbol{x}(t),\boldsymbol{y}(t))A(\boldsymbol{x}(t),t;\Delta t)\boldsymbol{g}(\boldsymbol{x}(t),\boldsymbol{y}(t)),\end{aligned} \tag{7}$$

where $\nabla_y\boldsymbol{h}$ denote the Jacobi matrix satisfying $(\nabla_y\boldsymbol{h})_{ij} = \partial h_i/\partial y_j$ and $A(\boldsymbol{x},t;\Delta t)$ is defined by

$$A(\boldsymbol{x},t;\Delta t) \equiv \int_t^{t+\Delta t}ds\int_t^s dt'\mathrm{e}^{-\frac{1}{\epsilon}\gamma(\boldsymbol{x})(s-t')}z_\epsilon(t')z_\epsilon(s). \tag{8}$$

With Eq. (3), the statistical average of Eq. (8) leads to

$$\begin{aligned}&\langle A(\boldsymbol{x},t;\Delta t)\rangle \\ &= \int_t^{t+\Delta t}ds\,(I+\gamma(\boldsymbol{x}))^{-1}\left[1-\mathrm{e}^{-\frac{1}{\epsilon}(I+\gamma(\boldsymbol{x}))(s-t)}\right] &(9)\\ &\to \Delta t\,(I+\gamma(\boldsymbol{x}))^{-1},\quad(\epsilon\to 0) &(10)\end{aligned}$$

where I denotes the unit matrix.

If we assume

$$\begin{aligned}&\langle z(t_1)z(t_2)z(t_3)z(t_4)\rangle \\ &\quad= \langle z(t_1)z(t_2)\rangle\langle z(t_3)z(t_4)\rangle(1+a\mathrm{e}^{-b(t_2-t_3)})\end{aligned} \tag{11}$$

for $t_1 \geq t_2 \geq t_3 \geq t_4$ with constants a and $b > 0$, which is satisfied by a wide class of noise such as the Ornstein-Uhlenbeck process and the dichotomous Markov noise[8], it is easily shown that

$$\langle [A_{ij}(\boldsymbol{x}, t; \Delta t)]^2 \rangle - \langle A_{ij}(\boldsymbol{x}, t; \Delta t) \rangle^2 \to 0 \quad (\epsilon \to 0) \tag{12}$$

holds. Equation (12) guarantees to replace $A(\boldsymbol{x}, t; \Delta t)$ in Eq. (7) by $\Delta t\,(I + \gamma(\boldsymbol{x}))^{-1}$ in the small limit of ϵ. Thus, by combining Eqs. (6) and (7) and noting that deviations of $\boldsymbol{y}(t)$ from 0 are small order terms, we obtain the Stratonovich type stochastic differential equation:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, 0) + \boldsymbol{h}(\boldsymbol{x}, 0)\xi + \nabla_y \boldsymbol{h}(\boldsymbol{x}, 0)\,(I + \gamma(\boldsymbol{x}))^{-1}\,\boldsymbol{g}(\boldsymbol{x}, 0), \tag{13}$$

where ξ denotes the Gaussian white noise[11] formally defined as $\lim_{\epsilon \to 0} z_\epsilon$ and satisfies

$$\langle \xi(t)\xi(s) \rangle = 2\delta(t - x). \tag{14}$$

3. Summary

The derivation of the reduced equation in the white noise limit for the system driven by colored noise is considered. It is shown that direct derivation of the reduced equation is possible without introducing the Fokker-Planck equation for the probability distribution or restricting to the cases of the Ornstein-Uhlenbeck process as the noise.

References

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[11] It is assumed that $z(t)$ and $z(s)$ are statistically independent in the limit of large $|t - s|$. This assumption may also justify Eq. (11) for the time correlation function.