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Multivariate extensions of recurrence networks reveal geometric signatures of coupling between nonlinear systems

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†*This contribution is dedicated to the commemoration of Jan H. Feldhoff, who performed most of the numerical studies in this work. On 28 February, 2014, Jan died all-to-early and unexpectedly after a fire in his home. We will remember him as an enthusiastic scientist and good friend.*

Abstract—Recurrence networks have recently proven their great potential for characterizing important properties of dynamical systems. However, in the real-world such systems typically do not evolve completely isolated from each other, but exhibit mutual interactions with their neighborhood. Here, we extend the recent view on isolated systems towards an coupled network approach to interacting systems. Specifically, we illustrate how to modify the concept of recurrence networks for studying dynamical interrelationships between two or more coupled nonlinear dynamical systems exclusively based on their attractors' geometric structures in phase space.

1. Introduction

In the last years, a variety of approaches has been proposed for studying time series by means of complex network methods [1]. As a particularly useful example, recurrence networks (RNs) [2, 3, 4] provide a widely applicable novel tool that has already proven its great potentials for geometrically characterizing dynamical systems and time series. Since RNs are based on a simple graph-theoretical re-interpretation of recurrence plots (RPs) [5], multivariate extensions of the RP concept provide prospective generalizations of RNs for studying interacting dynamical systems. Following this general idea, the present work reexamines cross- and as joint recurrence plots [5] from a complex network viewpoint. On the one hand, *inter-system recurrence networks* combine individual and cross-recurrence plots [6] and provide information on the coupling direction between two dynamical systems. Specifically, some structural characteristics of coupled networks display marked asymmetries that can be understood as signatures of directed couplings. On the other hand, *joint recurrence networks* [7] trace complex synchronization phenomena between coupled chaotic oscillators. The potentials and limitations of both approaches are briefly illustrated using two coupled Rössler oscillators as a paradigmatic model system.

2. Recurrence Networks

Recurrence plots (RPs) are an established tool for visualizing and quantitatively characterizing the temporal profile of recurrences of a complex system in its phase space. Specifically, RPs visulaize the proximity relationships between sampled state vectors $\{x_i\}$ in (the original or reconstructed) phase space of a system X encoded in the binary recurrence matrix [5]:

$$R^X_{ij}(\varepsilon) = \Theta(\varepsilon - \|x_i - x_j\|), \quad (1)$$

where ε is a threshold distance, and $\|\cdot\|$ denotes some norm (e.g., the Euclidean or maximum norm) in phase space.

Among other approaches, RPs can be used for analyzing time series by means of complex network concepts [2, 1, 3]. Specifically, for dissipative systems the resulting RNs' properties are uniquely determined by the invariant density and, hence, geometry of the underlying attractor, which also controls the system's dynamics. To obtain this RN representation, the RP of a time series is re-interpreted as the adjacency matrix of an undirected simple graph as $A_{ij}(\varepsilon) = R_{ij}(\varepsilon) - \delta_{ij}$ (where δ_{ij} is Kronecker's delta). The properties of RNs have been widely studied [3, 8, 9, 10], and their practical use as an exploratory tool of time series analysis has been demonstrated for a variety of model systems as well as real-world examples [2, 11, 12].

The transitivity properties of RNs provide useful measures for characterizing the effective local as well as global dimensionality of dynamical systems [9], since RNs are random geometric graphs with a vertex density given by the system's (invariant) density [4]. Specifically, the *transitivity dimension*

$$D_{\mathcal{T}} = \frac{\log \mathcal{T}(\varepsilon)}{\log(3/4)} \quad (2)$$

based on the network transitivity [13, 14]

$$\mathcal{T} = \frac{\sum_{i,j,k} A_{ij} A_{jk} A_{ki}}{\sum_{i,j,k} A_{ij} A_{ki}} \quad (3)$$

can be directly interpreted as a generalized fractal dimension [9].

3. Inter-system recurrence networks: Geometric inference of coupling directions

Generalizing the RP concept to a bivariate setting by studying the mutual proximity of state vectors of two dynamical systems sharing the same phase space, cross-recurrence plots (CRPs) [15] are defined as:

$$CR_{ij}^{XY}(\varepsilon) = \Theta(\varepsilon - \|x_i - y_j\|), \quad (4)$$

with x_i and y_i denoting points on some trajectories of two dynamical systems X and Y of lengths N_X and N_Y, respectively. In order to include the structure exhibited by a CRP into a complex network framework, RPs and CRPs are combined into one block-structured and symmetric *inter-system recurrence matrix* [6]

$$\mathbf{IR}(\boldsymbol{\varepsilon}) = \begin{pmatrix} \mathbf{R}^X(\varepsilon_X) & \mathbf{CR}^{XY}(\varepsilon_{XY}) \\ \mathbf{CR}^{YX}(\varepsilon_{YX}) & \mathbf{R}^Y(\varepsilon_Y) \end{pmatrix} \quad (5)$$

with $\mathbf{CR}^{XY} = [\mathbf{CR}^{YX}]^T$ being the cross-recurrence matrix between the systems X and Y and $\mathbf{R}^X = \mathbf{CR}^{XX}$ the recurrence matrix of system X (analogously for system Y). The latter definition can be directly generalized for $K > 2$ systems. Note that Eq. (5) possibly involves different recurrence thresholds. Specifically, it is recommended to chose these thresholds such that the auto- and cross-recurrence rates $RR_{X,Y} = [N(N-1)]^{-1}\sum_{i\neq j} R_{ij}^{X,Y}$ and $RR_{XY} = RR_{YX} = [N_X\ N_Y]^{-1}\sum_{i,j} CR_{ij}^{XY}$ fulfil the condition $RR_X = RR_Y > RR_{XY}$ [6]. Then, the adjacency matrix of the associated *inter-system recurrence network* (IRN) is $\mathbf{A}(\boldsymbol{\varepsilon}) = \mathbf{IR}(\boldsymbol{\varepsilon}) - \mathbb{I}_N$, where $\mathbb{I}_N$ denotes the N-dimensional unit matrix ($N = N_X + N_Y$).

Due to the specific choice of the block-wise (cross-) recurrence rates, the IRN has a distinct community structure separating the vertices representing the two systems under study. This allows viewing the IRN as two coupled networks [16]. For such coupled networks, the *cross-degree* k_i^{XY} counts the number of edges connecting a vertex i of system X to any vertex from system Y, i.e., $k_i^{XY} = \sum_j CR_{ij}^{XY}$. In a similar spirit, the *local cross-clustering coefficient* C_i^{XY} gives the probability that two randomly drawn neighbors j,k of vertex i from system X, which are both contained in the sample taken from system Y, are also neighbors:

$$C_i^{XY} = \begin{cases} \frac{\sum_{j\neq k} CR_{ij}^{XY} CR_{ik}^{XY} R_{jk}^{Y}}{k_i^{XY}(k_i^{XY}-1)}, & k_i^{XY} \geq 2 \\ 0, & \text{else.} \end{cases} \quad (6)$$

By averaging over all vertices i from system X, one obtains the corresponding *global cross-clustering coefficient* C^{XY}

$$C^{XY} = \frac{1}{N_X}\sum_i C_i^{XY}. \quad (7)$$

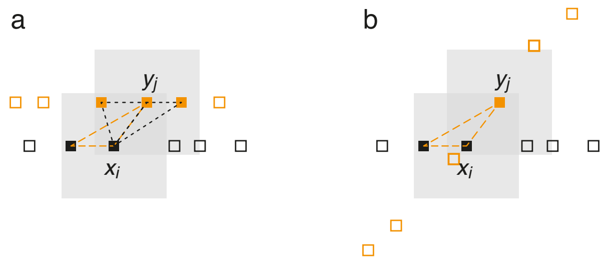

Figure 1: Schematic illustration of the neighborhoods of state vectors x_i and y_j in the case of (a) uncoupled and (b) unidirectionally coupled systems. Shaded areas represent neighborhoods of the respective state vectors, filled squares indicate close states. In case (b), the coupling increases the driven system's dimension, so that formerly close states are now outside of the neighborhood of y_j. Thus, the number of "cross-triangles" from X to Y decreases.

In a similar spirit, the network transitivity can be generalized to a coupled network measure, the cross-transitivity [16].

It is important to note that global cross-clustering coefficient and cross-transitivity are *not* invariant under the permutation $X \leftrightarrow Y$, i.e., $C^{XY} \neq C^{YX}$ and $\mathcal{T}^{XY} \neq \mathcal{T}^{YX}$. Given that X and Y represent qualitatively and quantitatively similar dynamics (e.g., two copies of the same dynamical system with only weak parameter mismatch), different cases can be distinguished: (i) In the *uncoupled* case, C^{XY} and C^{YX} "randomly" arise from the invariant densities of X and Y, so that one can expect $C^{XY} = C^{YX}$. (ii) For a *symmetric bidirectional* coupling or in a synchronized regime, the mutual effects on both systems are equal and thus lead to IRN measures of the same magnitude. (iii) For weak *unidirectional* coupling, the effective dimensionality of the driven system increases in comparison with the uncoupled case, since additional degrees of freedom are excited, and the dynamics of the driven system represents its internal variability plus the dynamics superimposed by the driver. This causes the number of recurrent states in the driven system to become reduced (Fig. 1b). As a consequence, if X drives Y, for two neighbouring states x_i and x_k in X it is more likely to also find a state y_j in Y that is close to both x_i and x_k, than in the opposite direction, which implies $C^{XY} < C^{YX}$.

To illustrate the latter conjecture, let us consider two diffusively coupled Rössler systems [17] in the funnel regime

$$\begin{aligned} \dot{x}^{(1)} &= -(1+\nu)x^{(2)} - x^{(3)} \\ \dot{x}^{(2)} &= (1+\nu)x^{(1)} + 0.2925x^{(2)} + \mu_{YX}(y^{(2)} - x^{(2)}) \\ \dot{x}^{(3)} &= 0.1 + x^{(3)}(x^{(1)} - 8.5) \\ \dot{y}^{(1)} &= -(1-\nu)y^{(2)} - y^{(3)} \\ \dot{y}^{(2)} &= (1-\nu)y^{(1)} + 0.2925y^{(2)} + \mu_{XY}(x^{(2)} - y^{(2)}) \\ \dot{y}^{(3)} &= 0.1 + y^{(3)}(y^{(1)} - 8.5), \end{aligned} \quad (8)$$

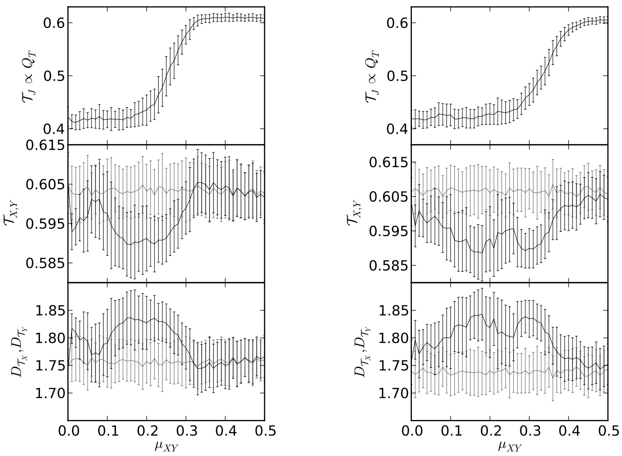

Figure 2: JRN transitivity $\mathcal{T}_J$, RN transitivities $\mathcal{T}_{X,Y}$, and corresponding transitivity dimensions $D_{\mathcal{T}_X}$, $D_{\mathcal{T}_Y}$ (from top to bottom) for two unidirectionally coupled Rössler systems ($X \to Y$) with $\nu = 0.02$ (driver oscillates faster than driven system, left panels) and $\nu = -0.02$ (right panels). Error bars indicate mean values and standard deviations estimated from 100 independent realizations ($N = 5,000$ data points) for each value of the coupling strength μ_{XY}. For RN transitivities and transitivity dimensions, gray (black) lines correspond to the values for system X (Y).

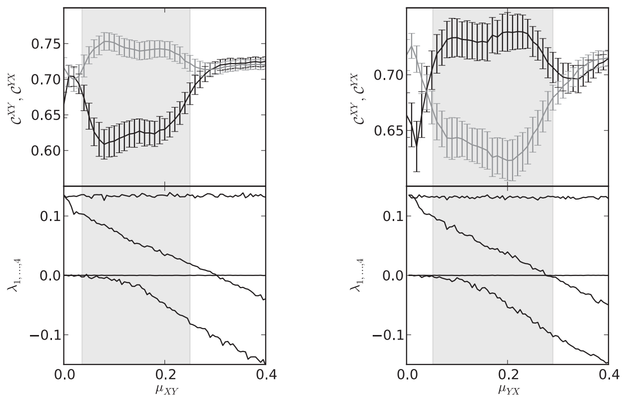

Figure 3: Global cross-clustering coefficients C^{XY} (black), C^{YX} (gray) and the four largest Lyapunov exponents for unidirectional coupling $X \to Y$ (left) and $Y \to X$ (right). The shaded regions mark the values of the coupling strength for which a correct identification of the coupling direction is achieved. Error bars represent mean values and standard deviations taken from an ensemble of 200 independent network realizations (with $N = 1,500$ data points per system).

which are completely identical besides a small detuning $\nu = 0.02$. In the uncoupled case ($\mu_{XY} = \mu_{YX} = 0$), the recurrence network transitivities of both systems are almost equal. For unidirectional coupling $X \to Y$ (i.e., $mu_{XY} > 0$, $\mu_{YX} = 0$), the transitivity dimension (2) allows tracing the difference in effective dimensionality between driving and driven system (Fig. 2). Moreover, the interacting network measures cross-transitivity and global cross-clustering coefficient display the expected behavior ($C^{XY} < C^{YX}$) for a wide range of moderate coupling strengths (Fig. 3) and only fail at very weak couplings (too small geometric signatures) and after the onset of synchronization (where both systems become dynamically undistinguishable).

4. Joint recurrence networks: Tracing generalized synchronization

The above generalization of CRPs to the network domain does not allow detecting generalized synchronization (GS), since cross-recurrences consider the co-occurrence of similar states, whereas GS implies that these states could be different. However, identifying GS in chaotic systems is still a subject of considerable interest. For this purpose, another complex network reinterpretation of a recurrence plot-based method can be applied. Specifically, *joint recurrence plots* (JRPs) are defined as the point-wise product of the RPs of two (or more) dynamical systems [18],

$$JR_{ij}^{XY}(\varepsilon_X, \varepsilon_Y) = R_{ij}^X(\varepsilon_X) \cdot R_{ij}^Y(\varepsilon_Y) \quad (9)$$

and have already been used for studying generalized synchronization [19]. Here $\varepsilon_{X,Y}$ are commonly chosen such that $RR_X = RR_Y$. Unlike CRPs, JRPs do not require the systems under study to share the same phase space, vut the sets of observation times have to be the same. Since JRPs obey the same symmetry as RPs, it is straightforward to define *joint recurrence networks* (JRNs) by setting $A_{ij} = JR_{ij} - \delta_{ij}$. Conceptually, JRNs can be interpreted in the same way as "normal" RNs in a higher-dimensional phase space jointly spanned by the relevant variables of both considered systems. However, one has to bear in mind that in the JRN, the distances are taken separately in the subspaces corresponding to the individual system's state spaces.

Since GS is related with an adjustment of the individual system's dynamics and, hence, dimensionality, taking the transitivity properties of the JRN into account provides valuable information about the onset of synchronization. Specifically, in the non-synchronized regime, the individual systems' RN transitivities $\mathcal{T}_{X,Y}$ are commonly much larger than that ($\mathcal{T}_J$) of the JRN in agreement with their interpretation as measures for the effective dimensionality. However, in the presence of GS, RN and JRN transitivities should take approximately the same value, i.e., the ratio $Q_{\mathcal{T}} = \mathcal{T}_J / \sqrt{\mathcal{T}_X \mathcal{T}_Y}$ converges to 1. Since $\mathcal{T}_{X,Y}$ do not change considerably as the coupling strength is increased (Fig. 2), the JRN transitivity $\mathcal{T}_J$ thus provides a useful indicator for the emergence of synchronization, which is applicable to both unidirectional as well as bidirectional coupling configurations [7].

5. Conclusions

This work has presented some novel approaches for generalizing the concept of recurrence networks to characterizing coupled dynamical systems. Specifically, inter-system recurrence networks (as a thorough extension of cross-recurrence plots) provide a useful tool for detecting univariate coupling from time series data. Unlike other existing methods serving the same purpose (see [6] for references), this approach is exclusively based on subtle geometric signatures of coupling resulting in a deformation of the driven system's attractor in its phase space. In turn, joint recurrence networks have demonstrated their capability of tracing the transition to generalized synchronization in coupled chaotic oscillators. A detailed exploration of the full potentials, as well as possible limitations of the proposed approaches will be the subject of future research.

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