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Comparizon between Spatiotemporally Discrete LCR Circuit Model and a Quantum Counterpart

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Abstract—Electric networks consisting of LCR elements are dissipative wave systems while quantum energy-conservative systems are dispersive wave systems. We compared wave propagation on spatiotemporally discretized models of the electric networks and on corresponding quantum systems. The two wave systems are parameterized with one common parameter. Although two kinds of propagation and their dependence on initial conditions are different because of lossy and lossless characteristics of the two systems, variances of the distributed waves depend similarly on the common parameter.

1. Introduction

A spatiotemporally discretized probabilistic cellular array model of LCR electric circuits was proposed. Single-electron digital filters [1, 2] and pseudorandom sequence generators [3] were built based on the architecture of the cellular array. In the cellular array, virtual charged particles move. Their directions of motion at time n depend on their directions at $n-1$. The probabilistic motion is described by a partial difference equation of a two-dimensional vector whose elements are probabilities that two particles moving from cells $i \pm 1$ exist at cell i.

Quantum walkers [4] are also described by a partial difference equation of two-dimensional vector referred to as spinor. Its difference from the vector expressing the probabilities of the particles in the cellular array is that its elements take complex values. The two elements express not only energy state but also spin orientation of the quantum walker. Each element evolves at time n depending on the two elements at time $n-1$

As mentioned above, the evolutions of the two vectors expressing the particles in the cellular array and the quantum walkers are similar. In this paper, the similar and different behaviors of the probabilistic particles and the quantum walkers are explored under a common parameter for the two partial difference equations. Difference of the dependence of the behaviors on the initial conditions are also investigated. In addition, continua of the two partial difference equations are compared and discussed.

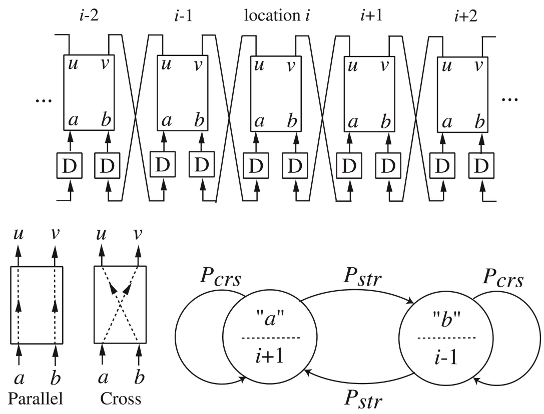

Figure 1: Cellular array model of LCR circuits. Upper: Structure of cellular array, Lower left: I/O connection inside cells, Lower right: Transition of particle position.

2. Cellular Array and Quantum Walker

2.1. Cellular Array

Figure 1 shows a cellular array model of LCR circuits. Each cell has two inputs a,b and two outputs u,v. Connections between the inputs and the outputs are in parallel or crossed at probabilities P_{str} and P_{crs} with $P_{str}+P_{crs} = 1$. Then, for each time step, the state of the D-flipflop connected to input "a" of the cell at i is shifted to the D-flipflop at input "b" of cell $i-1$ or at input "a" of cell $i+1$ at probabilities P_{str} and P_{crs} respectively, as shown in Fig. 1. By regarding the shifting state as a virtual particle moving in the cellular array, the following evolutionary equation of the existence probability of the particle is obtained:

$$a(n+1,i) = P_{crs}a(n,i-1) + P_{str}b(n,i-1) \tag{1}$$

$$b(n+1,i) = P_{str}a(n,i+1) + P_{crs}b(n,i+1) \tag{2}$$

where $a(n,i)$, $b(n,i)$ are respectively probabilities that the particle exists at inputs "a" and "b" of cell i at time n.

Let $\boldsymbol{p}(n,i)$ be defined as

$$\boldsymbol{p}(n,i) = [a(n,i) \;\; b(n,i)]^T \tag{3}$$

Then, Eqs. (1) and (2) are represented by

$$\boldsymbol{p}(n+1,i) = \boldsymbol{H}_U\boldsymbol{p}(n,i-1) + \boldsymbol{H}_L\boldsymbol{p}(n,i+1) \tag{4}$$

where matrices $\boldsymbol{H}_U$ and $\boldsymbol{H}_L$ are given by

$$\boldsymbol{H}_U = \begin{bmatrix} h_1 & h_2 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{H}_L = \begin{bmatrix} 0 & 0 \\ h_3 & h_4 \end{bmatrix}, \tag{5}$$
$$h_1 = h_4 = P_{cls}, \ \ h_2 = h_3 = P_{str}$$

The transition matrix for $\boldsymbol{p}(n, i)$,

$$\boldsymbol{H} = \boldsymbol{H}_U + \boldsymbol{H}_L \tag{6}$$

is a symmetric matrix with real elements, that is, an Hermitian matrix. Let P_{crs} and P_{str} be given by

$$P_{crs} = \cos^2\theta, \ \ P_{str} = \sin^2\theta \tag{7}$$

Then, matrix $\boldsymbol{H}$ is determined with a single-parameter θ.

2.2. Quantum Walk

States of quantum walkers, referred to as spinors, are expressed as

$$\boldsymbol{\psi}(n, i) = [\psi_u(n, i) \ \ \psi_d(n, i)]^T \tag{8}$$

Like Eq.(4) for the particle in the cellular array, the evolution of the spinor of a quantum walker is governed by the following equation with discrete independent variables of time n and space i:

$$\boldsymbol{\psi}(n + 1, i) = \boldsymbol{U}_U\boldsymbol{\psi}(n, i - 1) + \boldsymbol{U}_L\boldsymbol{\psi}(n, i + 1) \tag{9}$$

where $\boldsymbol{U}_U$ and $\boldsymbol{U}_L$ are given by

$$\boldsymbol{U}_U = \begin{bmatrix} u_1 & u_2 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{U}_L = \begin{bmatrix} 0 & 0 \\ u_3 & u_4 \end{bmatrix} \tag{10}$$

Their sum, the transition matrix for the spinor of a quantum walker,

$$\boldsymbol{U} = \boldsymbol{U}_U + \boldsymbol{U}_L \tag{11}$$

must be a unitary matrix since quantum walkers are quantum particles. The upper and lower entries of the spinor imply respectively spin orientation "up" and "down". Therefore, we find from Eq. (9) that the direction of the quantum walker depends on its spin orientation. Quantum walkers has been analyzed, for example, in [5, 6, 7, 8]. However, detailed analysis of time evolutions of their spin orientations has not yet been done.

By taking into account comparison with transition matrix $\boldsymbol{H}$ for the cellular array, one of the following matrices with parameter θ will be assigned as $\boldsymbol{U}$ for a quantum walker:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & \pm\boldsymbol{i}\sin\theta \\ \pm\boldsymbol{i}\sin\theta & \cos\theta \end{bmatrix}, \ \boldsymbol{i}^2 = -1 \tag{12}$$

Whichever matrix is chosen, the random walkers with appropriate initial state behave statistically equally after long time, which is realized by numerical experiments.

3. Numerical Experiments

3.1. Symmetric distribution

A cellular array and a quantum walk defined respectively by the following transition matrices $\boldsymbol{H}$ and $\boldsymbol{U}$ are compared:

$$\boldsymbol{H} = \begin{bmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{bmatrix}, \ \boldsymbol{U} = \begin{bmatrix} \cos\theta & -\boldsymbol{i}\sin\theta \\ -\boldsymbol{i}\sin\theta & \cos\theta \end{bmatrix}$$

Probability density functions that a particle in the cellular array and a quantum walker are at location i at time n are given respectively by

$$p(n, i) = a(n, i) + b(n, i) \tag{13}$$
$$p(n, i) = \boldsymbol{\psi}^*(n, i)^T\boldsymbol{\psi}(n, i) \tag{14}$$

Figure 2 shows the probability density functions for large n. The horizontal axes are $i/\sqrt{n}$ and i/n. In this figure, index k which determines parameter θ as

$$\theta = \frac{\pi}{2}\left(k - \frac{1}{2}\right), \ \ k = 1, 2, \cdots, 8. \tag{15}$$

is shown instead of θ. Initial states are set symmetric as given by

$$\boldsymbol{p}(0, i) = \begin{cases} \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}^T, & i = 0 \\ \begin{bmatrix} 0 & 0 \end{bmatrix}^T, & i \neq 0 \end{cases} \tag{16}$$

$$\boldsymbol{\psi}(0, i) = \begin{cases} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T, & i = 0 \\ \begin{bmatrix} 0 & 0 \end{bmatrix}^T, & i \neq 0 \end{cases} \tag{17}$$

For small k or θ, both probability density functions or $\boldsymbol{p}(n, i)$ and $\boldsymbol{\psi}(n, i)$ propagate like dissipation-less and dispersion-less waves, respectively. Then, their variances are larger. The propagation become slower with k or θ. Then, the variances become smaller, which is shown in Fig. 3.

3.2. Asymmetric distribution

Figure 4 shows probability density functions for the particle in the cellular array and the random walker when parameter θ is set to $\pi/4$ and initial states are set to

$$\boldsymbol{p}(0, i) = \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 \end{bmatrix}^T, & i = 0 \\ \begin{bmatrix} 0 & 0 \end{bmatrix}^T, & i \neq 0 \end{cases} \tag{18}$$

$$\boldsymbol{\psi}(0, i) = \begin{cases} \begin{bmatrix} 1/\sqrt{2} & \pm\boldsymbol{i}/\sqrt{2} \end{bmatrix}^T, & i = 0 \\ \begin{bmatrix} 0 & 0 \end{bmatrix}^T, & i \neq 0 \end{cases} \tag{19}$$

in addition to symmetric initial conditions (16) and (17). The three distributions for the particle in the cellular array are almost the same even if their initial distributions are different. On the other hand, distributions for the quantum walkers depend on their initial states. In other words, distributions for the quantum system leave forever marks of symmetry breaking at early time.

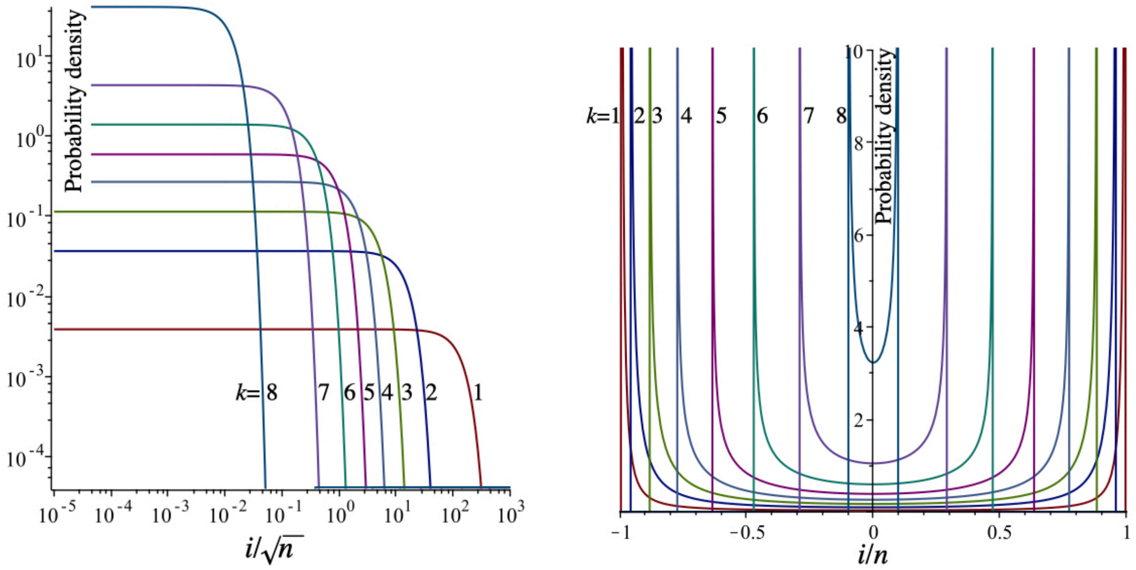

Figure 2: Probability density functions for cellular array (left) and quantum walk (right) with initial states (16) and (17).

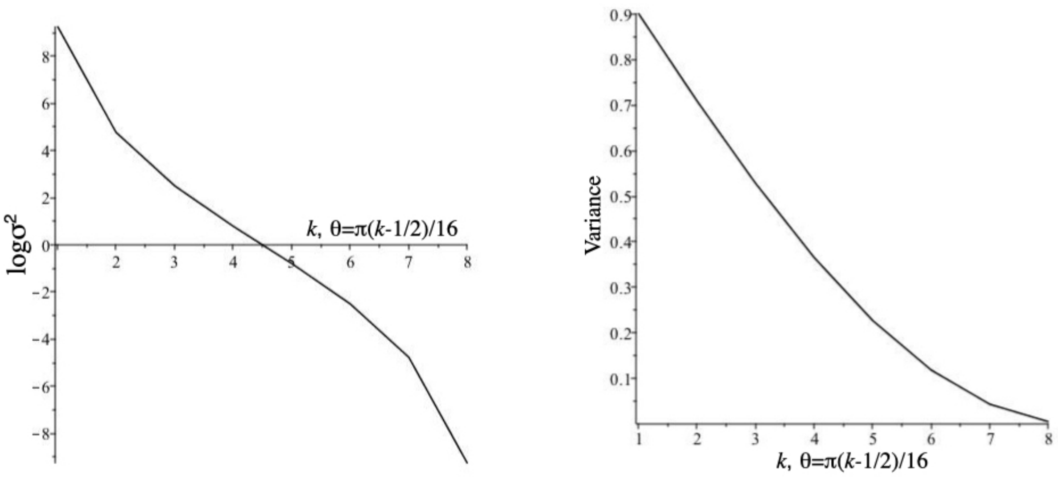

Figure 3: Variances of the Probability distributions for cellular array (left) and quantum walk (right).

4. Continua of Cellular Array and Quantum Walk

4.1. Continuum of Cellular Array

Spatial and temporal difference operators of the first and the second-order are defined as

$$\nabla_t a(n,i) \equiv a(n+1,i) - a(n,i) \tag{20}$$

$$\nabla_l a(n,i) \equiv a(n,i+1) - a(n,i) \tag{21}$$

$$\Delta_t a(n,i) \equiv \nabla_t a(n+1,i) - \nabla_t a(n,i) \tag{22}$$

$$\Delta_l a(n,i) \equiv \nabla_l a(n,i+1) - \nabla_l a(n,i) \tag{23}$$

By eliminating one of the two elements of $\boldsymbol{p}(n,i)$, for example, $b(n,i)$, a second-order partial difference equation of $a(n,i)$ is obtained. The equation is described with the operators in Eqs. (20) = (23) as

$$(2P_{crs} - 1)\Delta_t a(n,i+1) + 2(1 - P_{crs})\nabla_t a(n+1,i+1) = P_{crs}\Delta_l a(n+1,i) \tag{24}$$

Propagation of power waves $w(t,x)$ on a transmission line with series impedance $R + j\omega L$ and shunt admittance $j\omega C$ per unit length is described by

$$L\frac{\partial^2 w(t,x)}{\partial t^2} + R\frac{\partial w(t,x)}{\partial t} = \frac{1}{C}\frac{\partial^2 w(t,x)}{\partial x^2} \tag{25}$$

where x and t are the spatial and temporal independent variables. Partial differential equation (25) is the continuum corresponding to the discrete equation (24). Then, the cellular array is a model of lossless transmission lines when P_{crs}=1. The cellular array can be a model of drift-less diffusion system when P_{crs}=1/2.

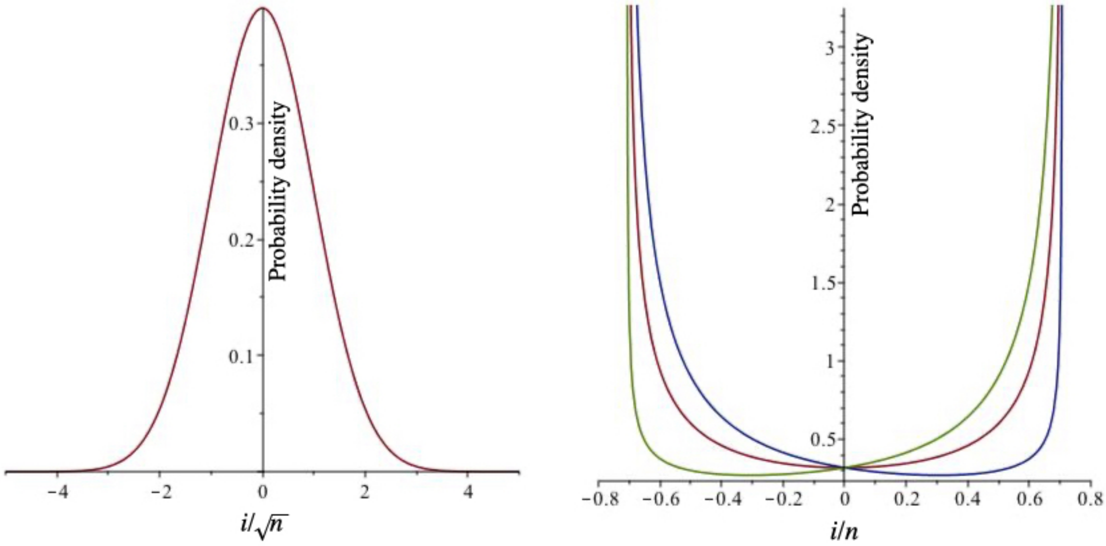

Figure 4: Probability density functions for cellular array (left, $\boldsymbol{p}(0,0)$ = [1, 0]:green, [1/2, 1/2]:red, [0, 1]:blue) and quantum walk (right, $\boldsymbol{\psi}(0,0) = [1/\sqrt{2}, -\boldsymbol{i}/\sqrt{2}]$:green, $[1/\sqrt{2}, 1/\sqrt{2}]$:red, $[1/\sqrt{2}, \boldsymbol{i}/\sqrt{2}]$:blue) with parameter $\theta = \pi/4$.

4.2. Continuum of Quantum Walk

Let a transition matrix for the quantum walk be given by

$$\boldsymbol{U} = \begin{bmatrix} \cos\theta & -\boldsymbol{i}\sin\theta \\ -\boldsymbol{i}\sin\theta & \cos\theta \end{bmatrix} \tag{26}$$

Continuum approximation by $x{=}\varepsilon i$, $t{=}\varepsilon n$, and $\varepsilon{\to}0$ leads Eq. (9) to the following equation:

$$\frac{\partial}{\partial t}\boldsymbol{\psi}(x,t) = \boldsymbol{\sigma}_z\boldsymbol{U}\frac{\partial}{\partial x}\boldsymbol{\psi}(x,t) + \{\boldsymbol{U}-\boldsymbol{E}\}\,\boldsymbol{\psi}(x,t), \tag{27}$$

$$\boldsymbol{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equation (27) is equivalent to Eq. (25) only if $\theta = 0$. In this case, both equations are wave equation with no dissipation nor dispersion.

5. Conclusions

We compared the motion of a particle on a spatiotemporally discretized cellular array model of LCR electric networks and a quantum walker on a quantum system. The two probabilistic systems are parameterized with one common parameter. It has been found that, although the two probability distributions in terms of the location of the particle and the walker are different because of lossy and lossless characteristics of the systems, the variances of the two distributions depend similarly on the common parameter. It has also been found that the asymmetry of initial probability distribution for the cellular array disappears while distribution for the quantum system leaves marks of symmetry breaking at early time.

The continuum for the difference equation describing the evolution of the distribution for the quantum walker was presented. The quantum system was equivalent to the LCR circuit only if parameter θ is zero.

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