Wave and Particle Models of Quantum Wave Lenses

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Abstract—In this paper, two-dimensional models of electron lenses, quantum systems with step and stair potentials are analyzed. Regarding electrons in the electron lenses both quantum waves and as probabilistic particles, we describe the behavior of the electrons by the Schrödinger and the Langevin equations. Solving the two equations, we found that the wave and particle models were equivalent. We analyzed numerically characteristics, for example, incidental versus refraction angles, of the lens model with step potential in detail and found that they were in accordance with theoretical characteristics.

1. Introduction

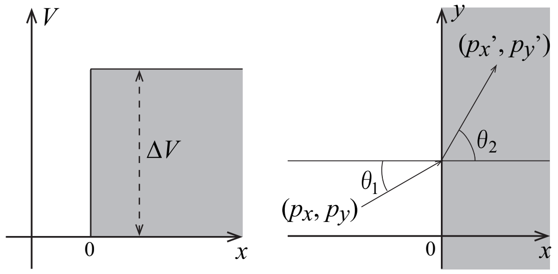
As we mentioned in [1], front-end filters are necessary between electromagnetic wave detectors based on photoelectric effect and single-electron tunneling (SET) electron-stream processors in terahertz sensing and communication systems.

The filters are used to remove photoelectrons excited by noise. They also may be applied to wavelength division multiple access communication systems. In conventional radar and wireless communication systems, distributed parameter filters are often used as front-end filters. Quantum periodic potential systems and quantum coupled wave guides correspond to stepped-impedance and coupled-line distributed parameter filters [1, 2].

On the other hand, in an optical system, a prism splits light into the spectrum of different colors. Quantum waves cause similar phenomenon in stepped potential systems. This refraction phenomenon can be applied to filtering electrons of a specified kinetic energy [3, 4].

In this paper, we investigate the quantum refraction in detail, specifically, incident versus refraction angles and potential height versus refraction angles. In addition, quantum refraction in a stair potential system is investigated in this paper. The stair potential has an advantage that the wave path can be shifted arbitrarily, which provides flexibility in physical circuit layout. In [1, 5], we mentioned the necessity of probabilistic particle models of electrons in electromagnetic wave detectors based on photoelectric effect and in quantum wave filters built by coupling wave guides. We also construct probabilistic particle models for the step and stair potential systems in this paper.

In this paper, physical units are defined as mentioned in [5]



(a) x-directional step potential (b) Incident and refraction angles

Figure 1: Potential of a model of electron lens with step potential.

2. Describing Step Potential System

We consider two-dimensional behavior of an electron in a potential $V(x, y) = V(x)$ which has step change ΔV in x-direction but is constant in y-direction as shown in Fig. 1. In a later numerical example, the step level change is set to $\Delta V = 10.0$. We assume that the wave function of the electron is in the following form:

$$\begin{aligned}\psi(x, y, t) &= \psi(x, t)\psi(y, t), \\ \psi(x, t) &= \phi(x)\exp\left(-i\frac{E_x}{\hbar}t\right), \\ \psi(y, t) &= \phi(y)\exp\left(-i\frac{E_y}{\hbar}t\right)\end{aligned} \tag{1}$$

Let (x_0, y_0) be the expectation of initial position of the electron and its momentum (p_x, p_y) be distributed initially as given by the following wave functions in momentum representation:

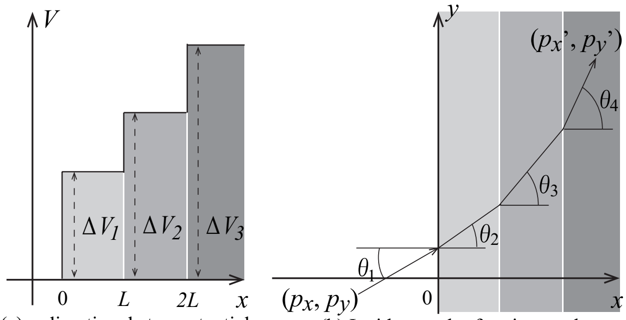
$$\phi_0(p_x) = \frac{1}{(2\pi\sigma_{p,x}^2)^{1/4}}\exp\left(-i\frac{p_x}{\hbar}x_0 - \frac{(p_x - p_{0,x})^2}{\sigma_{p,x}^2}\right) \tag{2}$$

$$\phi_0(p_y) = \frac{1}{(2\pi\sigma_{p,y}^2)^{1/4}}\exp\left(-i\frac{p_y}{\hbar}y_0 - \frac{(p_y - p_{0,y})^2}{\sigma_{p,y}^2}\right) \tag{3}$$

Then, wave functions $\psi(x, t)$ and $\psi(y, t)$ are given by

$$\psi(x, t) = \int_{-\infty}^{\infty}\phi_0(p_x)\phi(x)\exp\left(-i\frac{E_x}{\hbar}t\right)dp_x, \tag{4}$$

$$\phi(x) = \begin{cases} c_T\exp(i\frac{p_x}{\hbar}x) & \text{for } x \geq 0, \\ \exp(i\frac{p_x}{\hbar}x) + c_R\exp(-i\frac{p_x}{\hbar}x) & \text{for } x < 0, \end{cases}$$



(a) x-directional step potential (b) Incident and refraction angles

Figure 2: Potential of a model of electron lens with stair potential.

$$E_x = \begin{cases} p_x^2/2m + \Delta V & \text{for } x \geq 0, \\ p_x^2/2m & \text{for } x < 0, \end{cases}$$

$$\psi(y,t) = \int_{-\infty}^{\infty} \phi_0(p_y)\phi(y) \exp\left(-i\frac{E_y}{\hbar}t\right) dp_y, \tag{5}$$

$$\phi(y) = \exp(i\frac{p_y}{\hbar}y), \quad E_y = p_y^2/2m$$

Coefficients c_T and c_R are transmission and reflection rates for incidental electron waves at the potential wall at $x = 0$. An electron can exist in lower potential area at a probability close to 1 by determining appropriate expectation (x_0, y_0) at time $t = 0$. Initial momentum expectations are set to $p_{0,x} = p_0 \cos\theta_1$ and $p_{0,y} = p_0 \sin\theta_1$, where θ_1 is incident angle defined as shown in Fig. 1(b).

3. Describing Stair Potential System

We consider behavior of an electron in a stair potential shown in Fig. 2. The potential height is independent of y. It is expressed by

$$V(x) = \begin{cases} 0 & \text{for } x < 0 \\ \Delta V_1 & \text{for } 0 \leq x < L \\ \Delta V_2 & \text{for } L \leq x < 2L \\ \Delta V_3 & \text{for } 2L \leq x \end{cases} \tag{6}$$

In a later numerical example, $\Delta V_1 = 20.0$, $\Delta V_2 = 30.0$, $\Delta V_3 = 40.0$, and $L = 15$. The stair potential has the advantage that the electron path can be shifted in x-direction, which provides flexibility to circuit layout.

The wave function of a Schrödinger equation with potential (6) can be described by separable expression $\psi(x,t)\psi(y,t)$ as in Eq. (1). We assume that its initial wave distribution is given by Eqs. (2) and (3) in momentum representation. The following equation gives $\phi(x)$ when $\psi(x,t)$ is expressed in the same form with Eq. (4):

$$\phi(x) = \begin{cases} A_0 \exp(i\frac{p_x}{\hbar}x) + B_0 \exp(-i\frac{p_x}{\hbar}x) \\ \qquad\qquad \text{for } x < 0, \\ A_1 \exp(i\frac{p_x}{\hbar}x) + B_1 \exp(-i\frac{p_x}{\hbar}x) \\ \qquad\qquad \text{for } 0 \leq x < L, \\ A_2 \exp(i\frac{p_x}{\hbar}x) + B_2 \exp(-i\frac{p_x}{\hbar}x) \\ \qquad\qquad \text{for } L \leq x < 2L, \\ A_3 \exp(i\frac{p_x}{\hbar}x) \qquad \text{for } 2L \leq x \end{cases} \tag{7}$$

$$E_x = \begin{cases} p_x^2/2m & \text{for } x < 0, \\ p_x^2/2m + \Delta V_1 & \text{for } 0 \leq x < L, \\ p_x^2/2m + \Delta V_2 & \text{for } L \leq x < 2L, \\ p_x^2/2m + \Delta V_3 & \text{for } 2L \leq x \end{cases} \tag{8}$$

Coefficients A_i and B_i, $i = 0, 1, 2$, and 3, are determined by the continuity of $\phi(x)$ and $d\phi(x)/dx$ at $x = 0$, L, and $2L$. Since the potential is independent of y, $\psi(y,t)$ is described by Eq. (5).

4. Probabilistic Particle Model of the Electron

As the wave functions are determined for both the quantum systems with step and stair potentials, Nelson's stochastic quantization derives nonlinear Langevin equations whose solutions have the same probability distributions as those determined by the wave functions [5, 6]. The Langevin equation set is given by

$$\frac{dx(t)}{dt} = b_1(x,t) + \sqrt{\frac{\hbar}{2m}}\Gamma_1(t) \tag{9}$$

$$b_1(x,t) = \mathscr{R}[\chi_1(x,t)] + \mathscr{I}[\chi_1(x,t)], \tag{10}$$

$$\chi_1(x,t) = \frac{\hbar}{m}\nabla \ln \psi(x,t)$$

$$\frac{dy(t)}{dt} = b_2(y,t) + \sqrt{\frac{\hbar}{2m}}\Gamma_2(t) \tag{11}$$

$$b_2(y,t) = \mathscr{R}[\chi_2(y,t)] + \mathscr{I}[\chi_2(y,t)], \tag{12}$$

$$\chi_2(y,t) = \frac{\hbar}{m}\nabla \ln \psi(y,t)$$

$$< \Gamma_1(t)\Gamma_2(t') >= \delta(t - t') \tag{13}$$

where $\mathscr{R}[\chi]$ and $\mathscr{I}[\chi]$ are real and imaginary parts of χ respectively. The assertion means that a classical probabilistic lumped parameter system (9), (11) corresponding to a quantum system gradients of potential given by Eqs. (10), (12) and random noise of power $\hbar/2m$.

5. Numerical Analysis of Step Potential System

We obtained wave functions $\psi(x,t)$ for $t > 0$ when $\theta_1 = 30°$, $p_0 = 6.0$ and $\theta_1 = 60°$, $p_0 = 10.0$ initially. Variance of

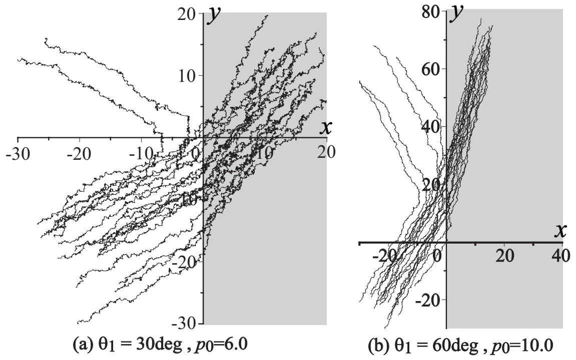


Figure 3: Sample trajectories of electrons in the lens model with step potential.

the momentum is set initially to $\sigma_{p,x}^2 = \sigma_{p,y}^2 = 0.01$. Figure 3 shows samples of numerical solutions of the Langevin equation describing a probabilistic particle model of the quantum step potential system. We see that several electrons reflect off the potential wall at $x = 0$ and other electrons penetrate into the higher potential area. Probability distribution in terms of the location of an electron is computed both from the wave function $\psi(x, t)$ and from the sample solutions of the Langevin equation. The two distributions obtained by the different two methods are almost equal as shown in Fig. 4.

By the energy conservation low in each direction, the relation between incident and refracting angles, θ_1, θ_2, depends on potential height and is given by [3]

$$\frac{\sin\theta_2}{\sin\theta_1} = \sqrt{\frac{(p_{0,x}^2 + p_{0,y}^2)/2m}{(p_{0,x}^2 + p_{0,y}^2)/2m - \Delta V}} \tag{14}$$

Figure 5 shows refraction angle θ_2 plotted against incidental angle θ_1 with parameter $\Delta V 2m/(p_{0,x}^2 + p_{0,y}^2) = 0.20$ and 0.13. Figure 6 shows refraction angle θ_2 versus step potential height ΔV when incidental angle θ_1 and kinetic energy $(p_{0,x}^2 + p_{0,y}^2)/2m$ are respectively 30° and 50.0 for one example and 30° and 75.0 for another example. Numerically obtained curves are computed from the sample trajectories of the probabilistic particle model constructed by Nelson's stochastic quantization. The curves are roughly in accordance with theoretical curves computed from Eq. (14).

6. Numerical Analysis of Stair Potential System

We execute numerical analysis of the electron behavior on the following conditions: The expectation of initial momentum is $p_0 = \sqrt{120}$, variance of the momentum is set initially to $\sigma_{p,x}^2 = \sigma_{p,y}^2 = 0.01$, and the incident angle θ_1 defined in Fig. 2(b) is 30°. We also analyzed electron behavior when the potential is a step function shown in Fig. 1 with $\Delta V = 40.0$ for comparison. The marginal probability distribution of x-directional location of an electron is

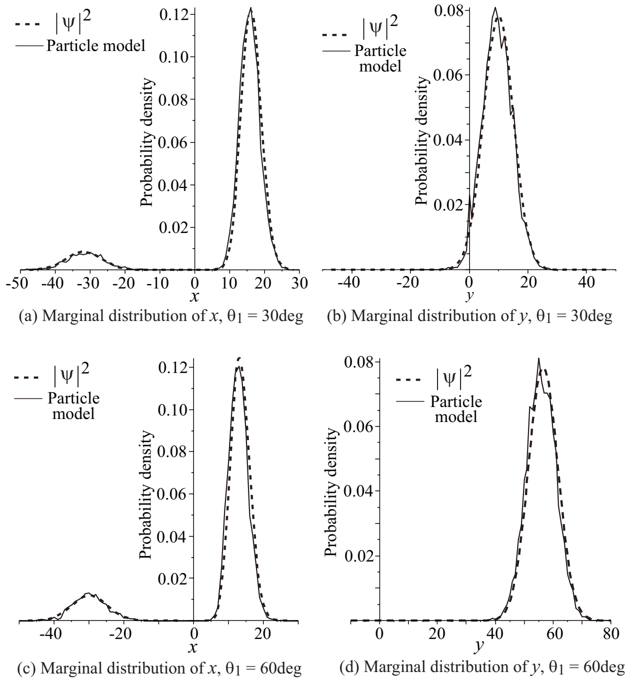


Figure 4: Probability distribution of the location of an electron in electron wave lens with step potential.

shown in Fig. 7. The distributions are obtained from the wave functions. Figure 8 shows the electron trajectories obtained from the Langevin equation derived by Nelson's stochastic quantization. Table 1 compares theoretical refracting angles and those estimated from the electron trajectories. We see that they are almost equal. We see also from Figs, 7, 8 and Tab. 1 that the electron path can be controlled by the potential form with refraction angle unchanged.

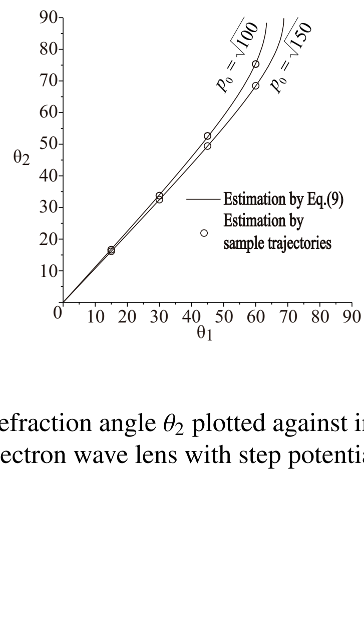


Figure 5: Refraction angle θ_2 plotted against incidental angle θ_1 for electron wave lens with step potential.

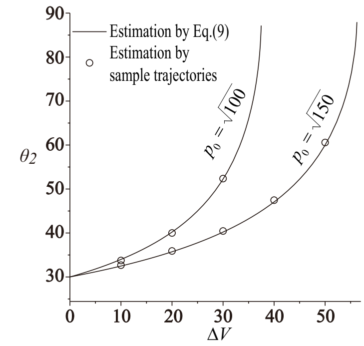
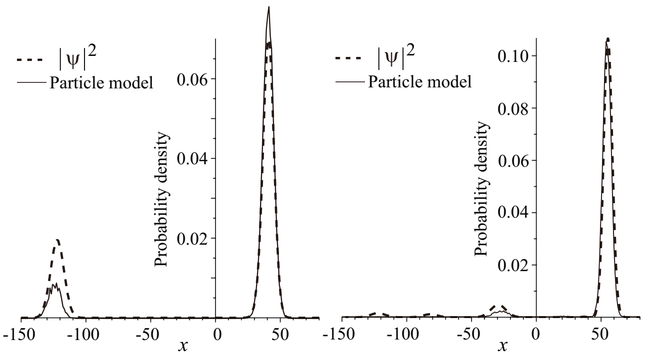


Figure 6: Refraction angle θ_2 plotted against step potential height ΔV for electron wave lens with step potential.

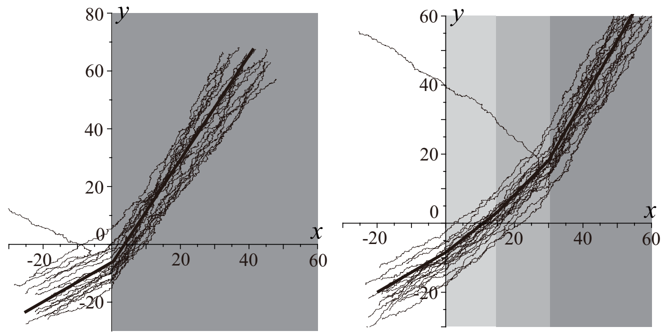


(a) Marginal distribution of x, step potential (b) Marginal distribution of x, stair potential

Figure 7: Marginal probability distribution of the x-directional location of an electron in the stair and step potentials.

7. Conclusions

In this paper, two-dimensional models of electron lenses, quantum systems with step and stair potentials were analyzed. The models' behavior obtained from the wave functions and from the behavior of the probabilistic particle models were almost equal. We analyzed the characteristics of the lens model with step potential in detail and found that they were in accordance with theoretical characteristics.



(a) Step potential (b) Stair potential

Figure 8: Sample trajectories of electrons in the stair and step potentials.

Table 1: Refracting angles.

Method	Angle θ_2 of reflaction in step potential (degree)	Angle of reflaction in stair potential (degree)		
		θ_2	θ_3	θ_4
Theoretical angles	60.0	37.8	45.0	60.0
Angles estimation by electron trajectories	60.6	40.7	44.8	60.1

References

[1] Nobuyuki Hirami, Hisato Fujisaka, and Takeshi Kamio, "Coupled Quantum Wave Guides," *Proc. of International Symposium on Nonlinear Theory and its Applications*, 2014.

[2] Nobuyuku Hirami, Hisato Fujisaka, and Takeshi Kamio "Probabilistic Particle Modeling of Quantum Wave Propagation with Reflection, Transmission, and Coupling," *Proc. of International Symposium on Circuits and Systems*,pp.478-481, 2014.

[3] U. Sivan, M. Heiblum, C. P. Umbach, and H. Shtrikman, "Electrostatic Electron Lens in the Ballistic Regime," *Physical Review B*, Vol. 41, Issue 11, pp.7937-7940, 1990.

[4] Yuma Kawabata, Hisato Fujisaka, and Takeshi Kamio 'Probabilistic Particle Modeling of Quantum Wave Propagation with Excitation and Refraction," *Proc. of International Symposium on Circuits and Systems*,pp.474-477, 2014.

[5] Takashi Murakami, Hisato Fujisaka, and Takeshi Kamio, "Wave and Particle Models of Single-Photoelectron Device," *Proc. of International Symposium on Nonlinear Theory and its Applications*, 2014.

[6] E. Nelson, "Derivation of the Schrödinger Equation from Newtonian Mechanics," *Physical Review*, Vol. 150, No. 4, pp.1079-1085, 1966.