

Bifurcation Analysis of a Chaotic Map with a Dynamic Controller Based on Stability Transformation Method

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Abstract—In this study, we consider a chaos control system based on a stability transformation method of unstable periodic orbits of a chaotic map. In our previous works, we have proposed a controlling method which transforms stability of unstable periodic orbits embedded on chaos attractor without the information of the location and shown a procedure to obtain the transformed system with stable periodic orbits which have same information of location to unstable periodic orbits on the original chaotic system. In this paper, we apply the method to unstable periodic orbits of logistic map and show procedure to design a continuous time system corresponding to the chaotic map with a controller based on the stability transformation. Also, the analysis of bifurcation of the system are provided.

1. Introduction

Nonlinear systems often exhibit unstable behavior called chaos that linear systems can not exhibit. In general, the procedure to stabilize unstable periodic orbits (abbr. UPOs) embedded on chaos attractor is called controlling chaos [1]. Many interesting works are reported for a control scheme of nonlinear systems which exhibits a chaos. On the other hand, the method called stability transformation [2] has been proposed in order to detect UPOs in dynamical systems which exhibit chaotic behaviors. The basic concept of stability transformation is to change the stability characteristic of the UPOs of chaotic systems by transformation based on chaos dynamics. Stability transformation is interesting not only for engineering applications, but also theoretical points of view. Also, stability transformation relates to delayed feedback control (abbr. DFC) method proposed by Pyragas [3]. DFC is said to a useful controlling chaos method which utilizes an information of the delay time, because DFC has advantages to stabilize UPOs without preliminary calculation of it. Thus, some interesting approaches to stabilize unknown stability characteristic have been proposed. In our previous works, we also have proposed a controlling method which transforms stability of unstable periodic orbits embedded on chaos attractor by using multirate sampled-data controllers [4]. Despite the proposed method is relatively simple, that makes it possible to stabilize UPOs that the information of the location is not known. We have discussed the stability of the system and the domain of attraction theoretically. However, the discussion of other parameter range which are not guaranteed the stability is inadequate and more consideration on wide parameter range is needed. In this work, we apply the method to unstable periodic orbits of logistic map and propose a continuous time system corresponding to the chaotic map with a controller based on the stability transformation. We provide the bifurcation analysis of the system.

2. A continuous time system corresponding to 1-dimensional chaotic map

2.1. Logistic map

In this paper, a target chaotic map is Logistic map. Logistic map is a difference equation that is obtained by discretizing the logistic equation which is a model of population growth [5]. The logistic map is described by

$$x_{n+1} = F_L(x_n) = ax_n(1 - x_n)$$

(0 < x_0 < 1, 0 < a \le 4). (1)

As well known the logistic map exhibits a chaos and its behavior is observed by iterating equation of (1).

2.2. Manifold piecewise linear chaos generator

Block diagram of a continuous time chaos generator corresponding to the logistic map is depicted in Fig. 1. This system is based on Manifold Piecewise Linear (abbr. MPL) system [6][7].

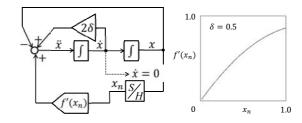


Figure 1: Block diagram of chaotic system.

The system consists of a linear amplifier, a block of nonlinear function, two integrator, and a sample and hold block. The triangle labeled 2δ is a linear amplifier with gain 2δ , the box labeled \int is an integrator, and the pentagon labeled $f'(x_n)$ is a time invariant block with the nonlinear function $f'(x_n)$. The block of S/H samples x when a

 $\dot{x}=0$ and stores the sampled value until next $\dot{x}=0$. The system dynamics is written by

$$\ddot{x} - 2\delta \dot{x} + x = f'(x_n) \quad (0 < \delta < 1),$$

$$x_n = x(kT) \quad \text{for} \quad kT \le \tau < (k+1)T,$$
(2)

where, $T=\frac{\pi}{\omega},\,kT$ is n-th moment of $\dot{x}=0,\,k$ is any positive integer, "is denotes differentiation by normalized time τ and δ is a damping parameter. An example of the trajectory of the system is depicted in Fig. 2. Supposing a state variable on the line of $\dot{x}=0$ is (x(0),0) at $\tau=0$, the trajectory started from the point moves around an equilibrium point (f'(x(0)),0) and must return to the line of $\dot{x}=0$ at $\tau=\frac{\pi}{\omega}$.

Adjusting the nonlinear function f', we can construct a continuous chaos generator corresponding to any 1-D return map.

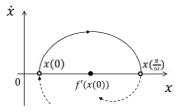


Figure 2: An Example trajectory on phase space.

In order to define return map, let $I = \{(x, \dot{x}) | \dot{x} = 0\}$. And letting any point on I be represented by its x-coordinate, x_n be the starting point and x_{n+1} be the return point, then we can define 1-D return map F.

$$F: x_n \mapsto x_{n+1}, x_{n+1} = F(x_n) = \alpha(x_n - f'(x_n)) + f'(x_n),$$
 (3)

where the α is constant number $-e^{\delta \frac{\pi}{\omega}}$. Here, the nonlinear function $f'(x_n)$ is defined by

$$f'(x_n) = \frac{1}{1-\alpha} (F_L(x_n) - \alpha x_n).$$

$$F_L(x_n) = ax_n (1 - x_n)$$
(4)

In this case, the return map F_L is a Logistic map. Using the system, we can construct a chaos generator corresponding to the logistic map. The simulated system behavior is shown in Fig. 3.

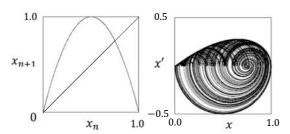


Figure 3: Return map and chaos attractor on phase space.

3. A continuous time system based on stability transformation

3.1. Controlling method based on stability transformation [4]

We show our controlling method that we have proposed that transforms stability of l-periodic orbits embedded on a chaos attractor. The dynamics of the transformed system is describe by

$$\begin{cases}
 x_{n+l} = F^{l}((1-K)x_{n} + Ky_{n}) & \text{for } n = kl, \\
 y_{n+l} = (1-K)x_{n} + Ky_{n} & \text{for } n = kl, \\
 x_{n+1} = F(x_{n}) & \text{for } n \neq kl, \\
 y_{n+1} = y_{n} & \text{for } n \neq kl,
\end{cases}$$
(5)

where (x_n, y_n) is a state of system, K is a control gain F is a chaotic map and has a fixed point $x_f = F(x_f)$.

At first, we consider a case of l=1 (i.e., fixed point). Substituting $x_n=F(y_n)$ into the (5), we can obtain the following:

$$y_{n+1} = (1 - K)F(y_n) + Ky_n, (x_n = F(y_n)).$$
 (6)

Accordingly, the condition for the stability of the system around the fixed point x_f is given by

$$\left| \frac{\partial y_{n+1}}{\partial y_n} \right|_{y_n = x_f} < 1. \tag{7}$$

Especially, if K is satisfied

$$K = \frac{a-2}{a-1},\tag{8}$$

the derivative $\frac{\partial y_{n+1}}{\partial y_n}$ at $y_n=x_f$ becomes to zero. That is, the fixed point of controlled system is super-stable on the condition (8). The stability transformation system (5) has a fixed point which has same information of the location to unstable fixed point on the original system. A fixed point on this system is stable when K is satisfied (7).

Next, we consider a case of l>1 (i.e., UPOs). We consider the stability of the system around a periodic point. p_i is defined as l-periodic point when $i=1,2,\cdots,l,$ $p_i=F^l(p_i)$ and $p_i\neq F^m(p_i)$, where $1\leq m< l$. If $|\frac{\partial F^l}{\partial x_n}|_{x_n=p_i}>1$ is satisfied, the l-periodic point p_i is unstable. Substituting $x_n=F^l(y_n)$ into the (5), We can obtain the following:

$$y_{n+l} = (1 - K)F^{l}(y_n) + Ky_n, (x_n = F^{l}(y_n)).$$
(9)

Thus, we can understand that this system transforms the stability of the fixed point of the composition map F_L^l . The condition for the stabilization of the system around the l-periodic point p_i is given by

$$\left| \left| \frac{\partial y_{n+l}}{\partial y_n} \right|_{y_n = p_i} \right| < 1. \tag{10}$$

Also, the slope of each periodic points of composition map F^l is defined as $A_{p_i} = \left. \frac{\partial F^l(x_n)}{\partial x_n} \right|_{x_n = p_i}$. The periodic point of control system is super-stable if K is satisfied

$$K = -A_{p_i}(1 - A_{p_i})^{-1}. (11)$$

The stability transformation system can stabilize *l*-periodic orbits which have same information of location to unstable periodic points on the original composition map. Typical shape of chaotic map and stability transformed map are is depicted in Fig. 4.

In the previous work, we have discussed only range of control gain K which can stabilizes l-periodic orbits (i.e., control gain K is satisfied (11)). In this paper, we consider wide parameter range of the control gain K. We note any control gain K and any parameter a of Logistic map. We analyze bifurcation behavior which occurs from stability transformation system.

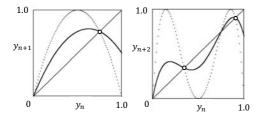


Figure 4: Typical shape of chaotic system and stability transformation system

3.2. A controller based on stability transformation

A continuous time system corresponding to the controlled map based on stability transformation is shown in Fig. 5.

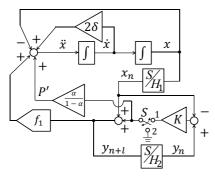


Figure 5: Block diagram of controller based on stability transformation

This system is constructed by based on dynamics (5). As with the chaos generator, the block of S/H_1 samples x when the $\dot{x}=0$ and stores the sampled value until next $\dot{x}=0$. The block of S/H_2 samples y_{n+l} when $\tau=klT(k=1,2,3\cdots)$ and stores the sampled value until $\tau=(k+1)lT$. The switch labeled S is defined by

$$S = \begin{cases} \text{"1"} & \text{for } klT \le \tau < (kl+1)T, \\ \text{"2"} & \text{otherwise.} \end{cases}$$
 (12)

The system dynamics is written by

$$\ddot{x} - 2\delta \dot{x} + x = \begin{cases} f_1 + P' & \text{for } klT \le \tau < (kl+1)T, \\ f_1 & \text{otherwise.} \end{cases}$$
(13)

Here, the nonlinear function f_1 and P' is defined by

$$f_1(y_{n+l}) = \frac{1}{1-\alpha} (F_L(y_{n+l}) - \alpha y_{n+l}), P' = \frac{K\alpha}{1-\alpha} (y_n - x_n).$$
(14)

This nonlinear function f_1 is same as the chaos generator. Using the system, adjusting control gain K, we can construct the control scheme corresponding to the controlled map based on stability transformation.

4. Bifurcation Analysis of the stability transformation system

In this section, we analyze bifurcation of the stability transformation system by numerical simulations. We consider that the system exhibits the bifurcation by two parameters a and K.

First, we consider the case of l=1. The bifurcation diagram of the system with l=1 is shown in Fig. 6.

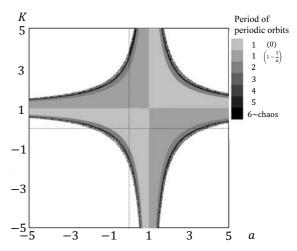


Figure 6: Bifurcation diagram (l = 1)

In Fig. 6, vertical axis is K, horizontal axis is a and color density of gray scale shows the period of periodic orbits. We can see that depending on values of each parameters, the system exhibits period-doubling bifurcation.

Next, we consider the case of l=2. The bifurcation diagram of the system with l=2 is shown in Fig. 7 and 8. In this case, depending on initial conditions, the attractors co-exist. Therefore, the system exhibits two kind of bifurcation diagrams as shown in Fig. 7 and 8. From these bifurcation diagrams, we can see that not only a region which exhibits period-doubling bifurcation depending on values of each parameters, but also a region which exhibits more complex bifurcation. The region which exhibits the complex bifurcation is shown in Fig. 9. The bifurcation diagram for control gain K is represent as shown in Fig. 10.

An example of return map and the corresponding attractor on the phase space of the controlled map is shown in Fig. 11. From Fig. 11, we can confirm that the system has stabilized two periodic orbits $\left\{-\frac{\sqrt{5}-5}{8}, \frac{\sqrt{5}+5}{8}\right\}$ embedded on a chaos attractor of logistic map in the condition for the stabilization $\left|\frac{\partial y_{n+2}}{\partial y_n}\right|_{y_n=p_i}$ < 1.

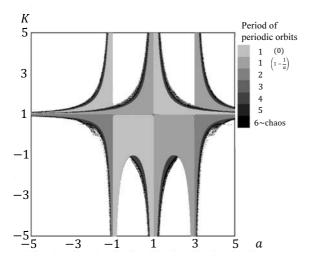


Figure 7: Bifurcation diagram (a) (l = 2)

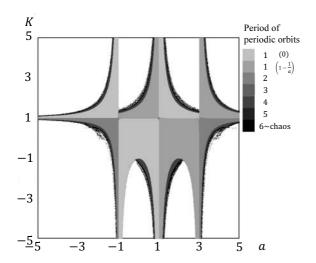


Figure 8: Bifurcation diagram (b) (l = 2)

5. Conclusions

In this paper, we consider a stability transformation method of unstable periodic orbits of a chaotic map. We showed procedure to design a continuous chaos generator corresponding to the logistic map and to the controlled map based on stability transformation. And, we provided the analysis of bifurcation of the system. In the future problems, we will analyze of bifurcation of the system in more detail, and verify the system behaviors in the experimental circuit.

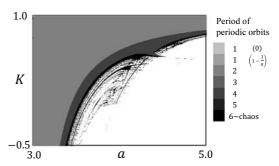


Figure 9: Enlarged bifurcation diagram (l = 2).

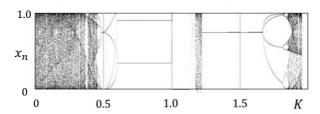


Figure 10: Bifurcation diagram (l = 2, a = 4).

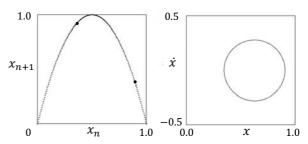


Figure 11: Return map and the attractor on phase space (l=2, a=4, K=0.8).

References

- E. Ott, C. Grebogi, and J. A. Yorke, "Controlling chaos," *Physical Review Letters*, vol. 64, no. 11, pp. 1196–1199, 1990.
- [2] D. Pingel, P. Schmelcher, and F. K. Diakonos, "Stability transformation: a tool to solve nonlinear problems," *Physics Reports*, 400, pp. 67–148, 2004.
- [3] K. Pyragas, "Continuous control of chaos by self-controlling feed-back," *Physical Letters A*, 170, pp. 421–428, 1992.
- [4] T. Tsubone, and N. Kambayashi, "Chaos and Periodic Oscillation of a Multirate Sampled-Data Dynamical System," *IEICE Ttrans. Fundamentals*, vol. E90-A, no.9, pp.1924–1931, 2007.
- [5] Robert. M. May, "Simple mathematical models with very complicated dynamics," *Nature*, vol. 261, pp. 459–467, 1976.
- [6] K. Mitsubori, and T. Sato, "Control of Piecewise Linear Chaos by Occasional Proportional Feedback," *Nonlinear dynamics:new theoretical and applied re*sults, pp. 361–375, 1995.
- [7] T. Tsubone, and T. Saito, "Stabilizing and Destabilizing Control for a Piecewise-Linear Circuit," *IEEE Trans. Circuits Syst. I*, vol. 45, no. 2, pp.172–177, 1998.
- [8] Y. Kawai, and T. Tsubone, "Composition of Stability Transformation System for Unstable Periodic Orbits," *IEICE Technical Report*, NLP2010-194 (2011-03), pp. 175–180, 2011.
- [9] T. Yoshioka, T. Tsubone, and T. Saito "On a Time-Variant Return Map of a Simple Switched Dynamical System," *IEICE Trans. A*, vol. J85-A, no. 8, pp. 908–914, 2002. (in japanese)