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A Class of Sinusoidal Driven Nonlinear Input-Output Systems with Sinusoidal Response

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Abstract—The Hopf amplifier based on the normal form equation of the Andronov-Hopf bifurcation is an established element in the modeling of the mammalian auditory system. Beside its specific amplification characteristic, it acts for single-tone signals like a linear system concerning the spectral behavior. Since this is a peculiar exception for a nonlinear amplifier, we use the Hopf normal form as a starting point to analyze all resonant two-dimensional differential equations with cubic nonlinearity. In particular we focus on the pure sinusoidal response.

1. Introduction

The detection and amplification of weak signals plays an important role in engineering sciences as well as in biology. For a variety of measurement systems beside RF and sensor applications a strong amplification of weak signals within a narrow bandwidth is required to get remarkable filtering characteristics in a noisy environment [1]. Furthermore, a high dynamic range is desirable to handle a wide range of signal levels [1]. The mammalian auditory system shows an excellent biological example for this task. It operates over an extraordinary range of input levels covering more than 120 dB [2]. Physiological measurements show, that this is achieved by a nonlinear dynamic compression, where larger amplifications occur towards lower input amplitudes and saturation towards strong forcings [3]. Moreover, decreasing the input amplitude is associated with a narrower bandwidth [3]. These insights motivate the modeling of the inner ear by using the normal form equation of the supercritical Andronov-Hopf bifurcation that shows qualitatively the behavior mentioned above [4]. To achieve the nonlinear amplification characteristic, the Hopf equation is endowed with a forcing term and tuned close to the onset of self-sustained limit cycle oscillations [4, 5]. Based on the suggested normal form of the Andronov-Hopf bifurcation, several models of the auditory system have been developed [5–7]. Despite the nonlinearity of the so-called Hopf amplifier, the study of the response characteristic has shown, that a sinusoidal input signal leads to a pure sinusoidal output signal with the same frequency and without any harmonic distortions [4–9]. Concerning the spectral behavior, this Hopf system acts like a linear system, which is a peculiar exception for a nonlinear amplifier [9]. Hence, this allows the calculation and characterization of the frequency dependent input-output amplitude and phase relations analytically [4, 5, 9]. Assuming harmonic distortions are undesirable, the mentioned behavior, besides the nonlinear amplification characteristic, provides an interesting basis for measurement systems or signal detectors. Since further equations are mentioned in this context [1, 10], we use the Hopf normal form as a starting point to analyze all systems belonging to the class of resonant two-dimensional differential equations with third-order nonlinearity. We present and discuss in our contribution a classification of the regarded systems, considering the input-output behavior, and in particular the sinusoidal response.

2. Class of Input-Output Systems

The (truncated) normal form equation of the Andronov-Hopf bifurcation is usually written in the complex form

$$\dot{z} = (\mu + \mathrm{i}\omega_0)\, z + \sigma\, |z|^2\, z, \quad z(t) \in \mathbb{C}, \tag{1}$$

where $\mu \in \mathbb{R}$ denotes the Hopf nonlinearity parameter, i is the imaginary unit and ω_0 is the natural frequency of the oscillation. In general, the coefficient σ is a complex quantity $\sigma = \sigma_R + \mathrm{i}\sigma_I$. Without loss of generality, our study is based on the ω_0-rescaled Hopf differential equation introduced by Stoop et. al. [5]

$$\dot{z} = \omega_0\,(\mu + \mathrm{i})\, z - \omega_0\, |z|^2\, z - \omega_0 f, \quad z(t), f(t) \in \mathbb{C}, \tag{2}$$

where the equation is extended by the forcing term $f(t)$, $\sigma = -1$ and $\mu < 0$. The conversion with $z(t) = x(t) + \mathrm{i}y(t)$ for the state variable and $f(t) = p(t) + \mathrm{i}q(t)$ for the forcing term leads to the real representation of (2)

$$\begin{aligned} \dot{x} &= -\omega_0 y + \mu\omega_0 x - \omega_0 x\left(x^2 + y^2\right) - \omega_0 p, \\ \dot{y} &= \quad \omega_0 x + \mu\omega_0 y - \omega_0 y\left(x^2 + y^2\right) - \omega_0 q. \end{aligned} \tag{3}$$

Since $z(t)$ and $f(t)$ are analytic signals, the imaginary parts $y(t)$ and $q(t)$ are the Hilbert transforms of the respective real parts $x(t)$ and $p(t)$. To classify the input-output behavior of all two-dimensional resonant parameter-dependent systems with cubic nonlinearity, we examine based on (3) all combinations described by

$$\begin{aligned}\dot{x} &= -\omega_0 y + a\mu\omega_0 x + \omega_0 \sum_{i=1}^{4} c_i x^k y^l - \omega_0 p, \\ \dot{y} &= \quad \omega_0 x + b\mu\omega_0 y + \omega_0 \sum_{j=1}^{4} d_j x^m y^n - \omega_0 q.\end{aligned} \tag{4}$$

Here, the coefficients of the linear damping are $\{a, b\} \in [\{1, 1\}, \{1, 0\}, \{0, 1\}]$ and of the cubic terms $c_i, d_j \in [-1, 0, 1]$ with $k+l = 3$ and $m+n = 3$, resulting in a set of 19680 equations. The calculation of the associated complex forms of the regarded systems (cf. (2)) allows to identify those systems where for $\mu < 0$ and $t \to \infty$ a complex sinusoidal output is a steady-state solution for a sinusoidal forcing. Despite the nonlinearity, the input-output amplitude and phase relations, which are parameter and frequency dependent, can be analytically specified for single-tone signals.

3. Systems with Sinusoidal Response

Our analysis of the large number of systems revealed, that only a few show a sinusoidal response to a sinusoidal forcing. Furthermore, it turned out that these systems can be classified into the following three categories:

- supercritical Hopf systems
- subcritical Hopf systems
- systems without Andronov-Hopf bifurcation

3.1. Supercritical Hopf Systems

Since the Hopf amplifier described by (2) provides the basis for our study, it exhibits the feature of a nonlinear dynamic compression with larger amplification towards weaker stimuli and an associated narrower bandwidth [5]. The algebraic equation describing this input-output behavior can be obtained by inserting the complex sinusoidal solution $z(t) = z_0 e^{i(\omega t+\varphi)}$ in (2), that results from the forcing $f(t) = f_0 e^{i\omega t}$. The evaluation of the squared modulus yields

$$f_0^2 = z_0^6 - 2\mu z_0^4 + \left[\mu^2 + (1 - \omega/\omega_0)^2\right] z_0^2. \tag{5}$$

Thus, the computation of the implicit equation (5) allows to depict the steady-state response of system (2) in dependency of the forcing amplitude f_0 and the nonlinearity parameter μ as shown in Fig. 1a),b). The mentioned behavior occurs with an increasing amplification of faint input signals while the μ-value is tuned closer to the bifurcation point. It should be noted, that the resonance peak always reaches its maximum at the characteristic frequency $\omega = \omega_0$. A more detailed description of the response characteristic of (2) can be found in [5].

Our analysis reveals in addition the following two supercritical Hopf systems with sinusoidal response to sinusoidal forcings

$$\dot{z} = \omega_0 (\mu + i) z - \omega_0 (1 - i) |z|^2 z - \omega_0 f, \tag{6}$$

$$\dot{z} = \omega_0 (\mu + i) z - \omega_0 (1 + i) |z|^2 z - \omega_0 f. \tag{7}$$

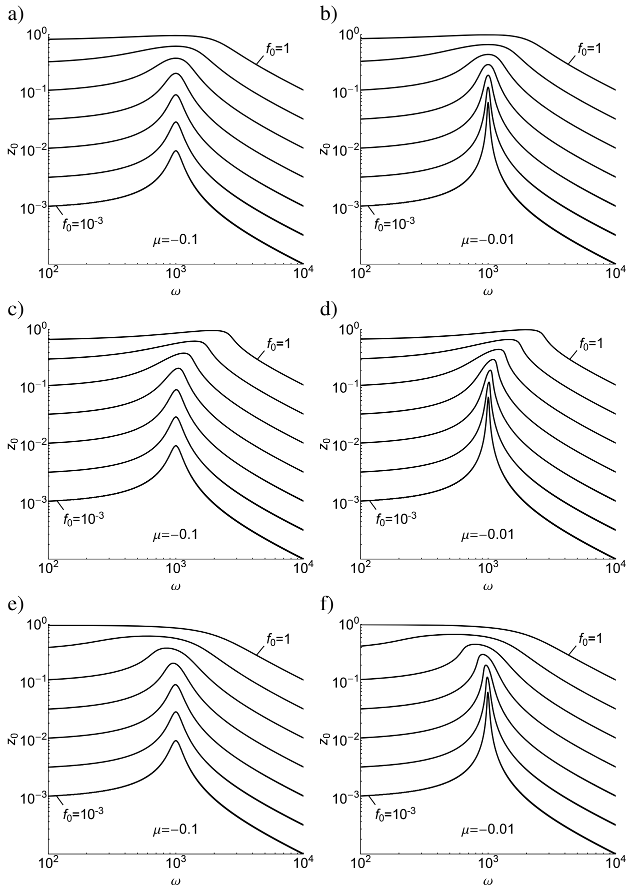

Figure 1: Steady-state responses; a),b) System (2) computed by (5); c),d) System (6) computed by (8); e),f) System (7) computed by (9); $\omega_0 = 10^3$

Here, in contrast to (2), the coefficient $\sigma = \sigma_R + i\sigma_I$ of the cubic term is complex. The relations of the input and output amplitudes of (6) and (7) can be calculated to

$$f_0^2 = z_0^6 - 2\mu z_0^4 + \left[\mu^2 + \left(1 - \omega/\omega_0 + z_0^2\right)^2\right] z_0^2, \tag{8}$$

$$f_0^2 = z_0^6 - 2\mu z_0^4 + \left[\mu^2 + \left(1 - \omega/\omega_0 - z_0^2\right)^2\right] z_0^2, \tag{9}$$

which lead to the response characteristics shown in Fig. 1c),d) and Fig. 1e),f), respectively. We can deduce that for stronger stimuli the resonance peak shifts with $\sigma_I = 1$ slightly to higher frequencies, and with $\sigma_I = -1$ towards lower frequencies. We emphasize, the latter effect also exists in the experimentally obtained response characteristics of the inner ear [4]. Thus, some authors argue to use Hopf systems with a complex cubic term for modeling the auditory system [7]. In contrast, it has been demonstrated that a chain of feed-forward coupled Hopf amplifiers described by (2), alternating with low-pass filters, also reproduces the shape of the auditory response curves with the desired left shift of the resonance peak [5]. Moreover, the systems show for lower input amplitudes a μ-dependent amplification behavior of the same kind as system (2).

3.2. Subcritical Hopf Systems

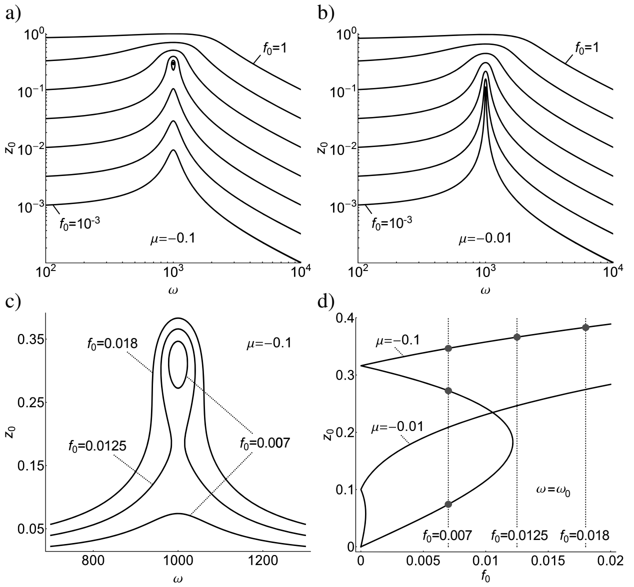

Figure 2: a),b) Steady-state responses of system (10) computed by Eq. (13); c),d) Ambiguity in the response curves for $\mu = -0.1$ and selected values of f_0; $\omega_0 = 10^3$

Further investigations of the systems revealed the subcritical counterparts of the supercritical Hopf systems treated before.

$$\dot{z} = \omega_0 (\mu + \mathrm{i}) z + \omega_0 |z|^2 z - \omega_0 f, \tag{10}$$

$$\dot{z} = \omega_0 (\mu + \mathrm{i}) z + \omega_0 (1 + \mathrm{i}) |z|^2 z - \omega_0 f, \tag{11}$$

$$\dot{z} = \omega_0 (\mu + \mathrm{i}) z + \omega_0 (1 - \mathrm{i}) |z|^2 z - \omega_0 f. \tag{12}$$

The evaluation of the associated input-output relations,

$$f_0^2 = z_0^6 + 2\mu z_0^4 + \left[\mu^2 + (1 - \omega/\omega_0)^2\right] z_0^2, \tag{13}$$

$$f_0^2 = z_0^6 + 2\mu z_0^4 + \left[\mu^2 + \left(1 - \omega/\omega_0 + z_0^2\right)^2\right] z_0^2, \tag{14}$$

$$f_0^2 = z_0^6 + 2\mu z_0^4 + \left[\mu^2 + \left(1 - \omega/\omega_0 - z_0^2\right)^2\right] z_0^2, \tag{15}$$

revealed overall the same behavior as the supercritical Hopf systems in Sec. 3.1. As an example for these systems, we illustrate the steady-state response of (10) described by (13) in Fig. 2a),b). Due to the imaginary part of σ, the systems (11) and (12) show an equivalent shift of the resonance peak to higher and lower frequencies as the systems (6) and (7) in Fig. 1c),d) and Fig. 1e),f), respectively. However, for certain relations of the parameter μ and the forcing amplitude f_0, ambiguities in the response curves arise (cf. Fig. 2c),d)). This effect has also been observed and discussed for a driven supercritical Andronov-Hopf equation [11]. Since the presented systems exhibit a subcritical Hopf bifurcation ($\sigma_R > 0$), an unstable limit cycle restricts the forcing amplitude and the initial conditions. Hence, these systems appear impractical as nonlinear amplifier.

3.3. Systems without Hopf Bifurcation

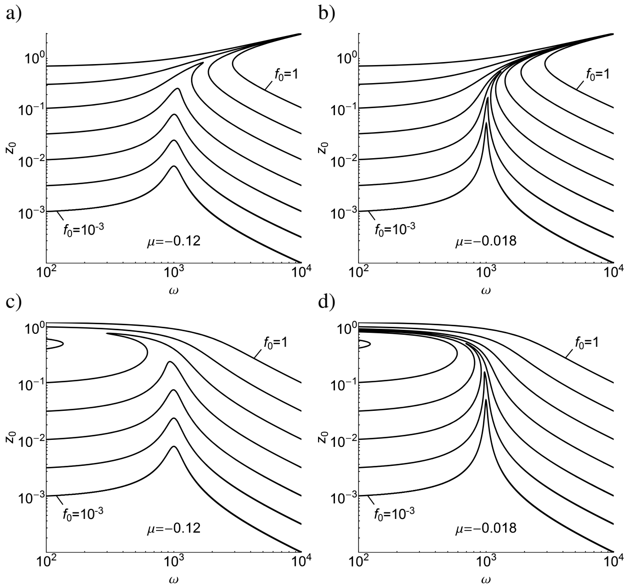

Figure 3: a),b) Steady-state responses of system (16) computed by Eq. (18); c),d) Steady-state responses of system (17) computed by Eq. (19); $\omega_0 = 10^3$

Our study reveals systems with a sinusoidal response to a sinusoidal forcing, that are not exhibiting any Hopf bifurcation. This two systems are

$$\dot{z} = \omega_0 (\mu + \mathrm{i}) z + \omega_0 \mathrm{i} |z|^2 z - \omega_0 f, \tag{16}$$

$$\dot{z} = \omega_0 (\mu + \mathrm{i}) z - \omega_0 \mathrm{i} |z|^2 z - \omega_0 f. \tag{17}$$

Obviously, the equations have a purely imaginary coefficient σ_I of the cubic term. The respective input-output behavior, calculated by

$$f_0^2 = \left[\mu^2 + \left(1 - \omega/\omega_0 + z_0^2\right)^2\right] z_0^2, \tag{18}$$

$$f_0^2 = \left[\mu^2 + \left(1 - \omega/\omega_0 - z_0^2\right)^2\right] z_0^2, \tag{19}$$

is illustrated in Fig. 3. Similar to the systems presented before, the resonance peak shifts for strong forcings towards higher and lower frequencies in dependency of the sign of σ_I. However, the shift is much stronger leading to ambiguity and hysteretic behavior in the response curves. In contrast to the other studied systems, an exception to the output amplitude z_0 being limited for $f_0 = 1$ occurred in system (16). Furthermore, for weak input signals, the systems display a similar μ-dependent amplification characteristic as mentioned for the regarded Hopf systems in Sec. 3.1 and 3.2.

4. Restricted Systems with Sinusoidal Response

Beside the systems presented in Sec. 3, that always lead in steady-state to sinusoidal responses to sinusoidal forcings, our study reveals systems that show this behavior merely for the restricted forcing frequency $\omega = \omega_0$. Since these systems exhibit both a supercritical and a subcritical Andronov-Hopf bifurcation, we will refrain from discussing the latter one. The systems displaying a supercritical bifurcation are formulated as

$$\dot{z} = \mathrm{i}\omega_0 z + \frac{\omega_0\mu}{2}(z + \bar{z}) - \frac{\omega_0}{2}|z|^2(z + \bar{z}) - \omega_0 p, \quad (20)$$

$$\dot{z} = \mathrm{i}\omega_0 z + \frac{\omega_0\mu}{2}(z - \bar{z}) - \frac{\omega_0}{2}|z|^2(z - \bar{z}) - \mathrm{i}\omega_0 q, \quad (21)$$

where $\bar{z}$ is the complex conjugate of z. From the more accessible real representations of (20) and (21)

$$\begin{aligned} \dot{x} &= -\omega_0 y + \mu\omega_0 x - \omega_0 x\left(x^2 + y^2\right) - \omega_0 p, \\ \dot{y} &= \quad \omega_0 x, \end{aligned} \quad (22)$$

$$\begin{aligned} \dot{x} &= -\omega_0 y, \\ \dot{y} &= \quad \omega_0 x + \mu\omega_0 y - \omega_0 y\left(x^2 + y^2\right) - \omega_0 q, \end{aligned} \quad (23)$$

it is apparent that these systems are variants of the driven Rayleigh-van der Pol equation [12]. Since the sinusoidal forcing $f(t) = f_0 e^{\mathrm{i}\omega_0 t} = p(t) + \mathrm{i}q(t)$ is an analytic signal, it is obvious that the input signals of (20) and (21), $p(t) = f_0 \cos(\omega_0 t)$ and $q(t) = f_0 \sin(\omega_0 t)$, are real sinusoidal signals, that lead to the complex solution $z(t) = z_0 e^{\mathrm{i}(\omega_0 t + \varphi)}$. For both systems, the relation between the input amplitude f_0 and the output amplitude z_0 can be obtained by $f_0 = \mu z_0 - z_0^3$. This describes exactly the same amplification characteristic as system (2) for $\omega = \omega_0$ (cf. (5)). For input signals with $\omega \neq \omega_0$ harmonic distortions emerge due to the nonlinearity of the systems (cf. [9]).

5. Conclusion

In this paper we analyzed all systems comprised by the class of resonant two-dimensional differential equations with third-order nonlinearity. We have shown, that only systems described by the following driven normal form equation of the Andronov-Hopf bifurcation

$$\dot{z} = \omega_0(\mu + \mathrm{i})z + \omega_0\sigma|z|^2 z - \omega_0 f$$

act like a linear amplifier with respect to the spectral behavior. For a sinusoidal input signal, they respond with a pure sinusoidal output signal without any harmonic distortions. In dependency of $\sigma = \sigma_R + \mathrm{i}\sigma_I$ the systems can be categorized into supercritical Hopf systems with $\sigma_R < 0$, their subcritical counterparts with $\sigma_R > 0$, and systems without any Andronov-Hopf bifurcation $\sigma_R = 0$. For $\sigma_I = 0$ the resonance peaks in the response curves reach their maximum always at the characteristic frequency $\omega = \omega_0$. Otherwise the peak is shifted towards lower ($\sigma_I < 0$) or higher ($\sigma_I > 0$) frequencies for strong forcings. This can lead to ambiguities in the response curves. The latter effect can also be observed for subcritical Hopf systems with $\sigma_I = 0$. Since they exhibit an unstable limit cycle, these systems appear impractical as nonlinear amplifiers. Our study revealed that the Rayleigh-van der Pol equation shows a sinusoidal response for the restricted frequency $\omega = \omega_0$ of the single-tone forcing. All other systems with cubic nonlinearity, like the van der Pol equation, that has been used as a signal detector [1], lead always to harmonic distortions (cf. [9]). Especially systems with more than one real fixed point (e.g. [10]) show large distortions or even instable solutions.

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