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Investigation of Itinerancy of Distance between Solution Trajectories Including Phase-Inversion Waves by Slightly Different Two Initial Values on a Torus of Coupled Van der Pol Oscillators

Kyosuke YASUDA†, Keisuke NAGAI†, and Masayuki YAMAUCHI‡

†Hiroshima Institute of Technology
2–1–1 Miyake, Saeki-ku, Hiroshima, 731–5193, Japan
Email: ma23020@cc.it-hiroshima.ac.jp

Abstract— We have been investigating various synchronization phenomena observed on coupled oscillator networks. In this paper, we confirm the phase-inversion waves can be observed on a network which is constructed by coupled 10×10 van der Pol oscillators as a torus. Furthermore, we investigate behavior of which distance between two solution trajectories of instantaneous electric powers by very small difference between two initial values and consider that the synchronization phenomena with phase-inversion waves are stable solution or not.

1. Introduction

In this world, there are many synchronization phenomena of chemical reactions, physical phenomena, and so on. We cannot live without synchronization phenomena, because it is considered that a heart move depending on the pacemaker cells which synchronizes. Synchronization phenomena can be observed on many electric and electronic circuits, and the phenomena are used for industrial products. For examples, communication system can be not built without synchronization phenomena. Therefore, it is very important for scientific development that the synchronization phenomena are investigated and analyzed.

Synchronization phenomena on a circuit of which many van der Pol oscillators are coupled as a ladder, a 2d-lattice, and a ring, have been investigated and analyzed[1]-[3]. We have been investigating and analyzing phase-inversion waves on the ring, the ladder, the 2d-lattice, and so on[4]-[5]. These phase-inversion waves cannot be theoretically proved one of stable solution, yet. Therefore, we have been developing new analyzing method for proving stable solution or not. It is determining methods based on whether the differences between two sets of initial values are expanded or not.

In this study, we confirm that the phase-inversion waves can exist on the torus of which 10×10 van der Pol oscillators are coupled by inductors. Next, we investigate difference between depending two solution trajectories on two slightly different initial values and discuss stability, when the phase-inversion waves are propagating on the torus. In this investigation, we use an instantaneous electric power in each oscillator.

ORCID iDs First Author: 0009-0001-1404-7649, Second Author: 0009-0002-6913-4915, Third Author: 0000-0002-6485-8298

2. Circuit model

A circuit model of this study is shown in Fig. 1. This figure shows a torus of 10×10 van der Pol oscillators. The torus is constructed by which 10×10 van der Pol oscillators are coupled by inductors as a 2d-lattice, and top side and bottom side, and left side and right side are coupled, respectively. A nonlinear negative resistor is used in each oscillator. An approximate equation of the nonlinear negative resistor in the $\mathrm{OSC}_{i,j}$ is shown as Eq. (1).

$$f(v_{i,j}) = -g_1 v_{i,j} + g_3 v_{i,j}^3 \quad (g_1,\ g_3 > 0). \tag{1}$$

<Normalized circuit equations>

Circuit equations of $\mathrm{OSC}_{i,j}$ is normalized by Eq. (2), and shown in Eq. (3).

$$\begin{aligned} & i_{i,j} = \sqrt{\frac{Cg_1}{3Lg_3}} x_{i,j}, \quad v_{i,j} = \sqrt{\frac{g_1}{3g_3}} y_{i,j}, \quad t = \sqrt{LC}\tau, \\ & \alpha = \frac{L}{L_0}, \quad \varepsilon = g_1 \sqrt{\frac{L}{C}}, \quad \delta = \frac{g_1^2}{3g_3}. \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{dx_{i,j}}{d\tau} &= y_{i,j}, \\ \frac{dy_{i,j}}{d\tau} &= -x_{i,j} + \varepsilon\left(y_{i,j} - \frac{1}{3}y_k^3\right) \\ &\quad + \alpha\left(x_{i-1,j} + x_{i,j-1} + x_{i,j+1} + x_{i+1,j} - 4x_{i,j}\right). \end{aligned} \tag{3}$$

The α shows coupling strength, and the ε expresses nonlinearity. In this study, we calculate these equations by using fourth order Runge-Kutta method and a step size of calculation is fixed as $0.0001[\tau]$.

<Normalized instantaneous electric power>

A normalized instantaneous electric power $p_{i,j}$ is shown as Eq. (4).

$$p_{i,j} = \frac{\alpha\delta}{\varepsilon} y_{i,j}\left(x_{i-1,j} + x_{i,j-1} + x_{i,j+1} + x_{i+1,j} - 4x_{i,j}\right) \tag{4}$$

The δ shows as magnification ratio of power. Therefore, we set δ as 1 in this research.

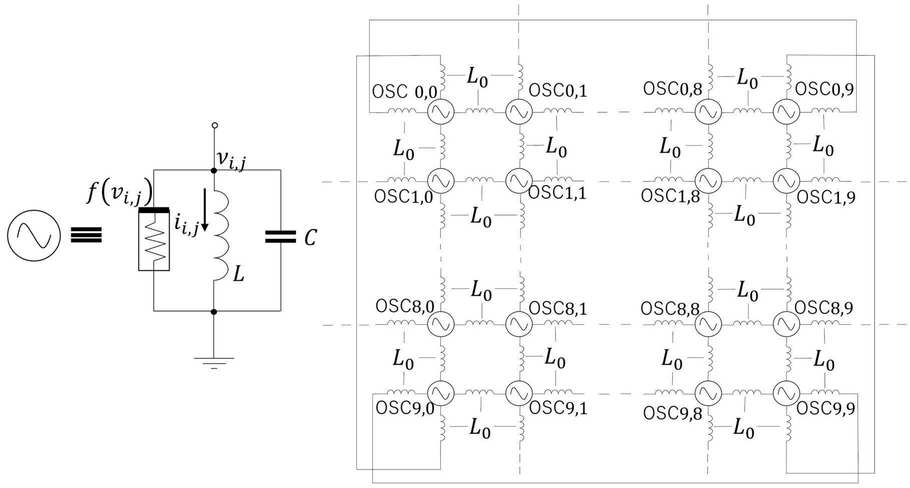

Figure 1: Circuit model of a torus of 10×10 van der Pol oscillators.

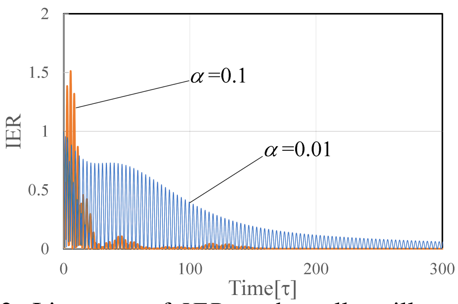

Figure 2: Itinerancy of $IER_{1,1}$ when all oscillators are in-phase synchronization and $\varepsilon = 0.05$ (blue line: $\alpha = 0.01$, and orange line: $\alpha = 0.1$).

<Instantaneous expanding rate>

We investigate that distance between two solution trajectories by two sets of initial values becomes large or not, because we want to investigate stability of the system with phase-inversion waves. If the distance becomes small, we can guess that the solution is stable.

Two sets of initial values with slightly different are input the circuit model, The two sets of initial values are named initial-value-set A and initial-value-set B, respectively. In this study, $x_{1,1}$ or $y_{1,1}$ of initial-value-set B is 0.001 larger than $x_{1,1}$ or $y_{1,1}$ of initial-value-set A, respectively. An instantaneous power $p_{i,j}$ of m-th calculation result of initial-value-set A is named $p_{Ai,j}(m)$, and $p_{i,j}$ by initial-value-set B is shown $p_{Bi,j}(m)$. A distance between $p_{Ai,j}(m)$ and $p_{Bi,j}(m)$ is expressed as $DP_{i,j}(m)$ in Eq. (5), and we define "Instantaneous Expanding Rate($IER_{i,j}(m)$)" as expanding rate for an initial distance $DP_{i,j}(0)$. The $IER_{i,j}(m)$ is shown as Eq. (6).

$$DP_{i,j}(m) = \left| p_{Ai,j}(m) - p_{Bi,j}(m) \right| \tag{5}$$

$$IER_{i,j}(m) = \frac{DP_{i,j}(m)}{DP_{i,j}(0)} \tag{6}$$

The itinerancy of $IER_{1,1}$ is shown in Fig. 2 of when all oscillators are synchronizing as an in-phase, and values of only $OSC_{1,1}$ of initial-value-set B are slightly changed from initial-value-set A. The vertical axis shows value of the IER, and the horizontal axis is time. We can confirm that $IER_{1,1}$ become almost 0, and all van der Pol oscillators become stable in the in-phase synchronization, and the convergence speed of when α is 0.01 is faster than the convergence speed of when α is 0.001.

3. Synchronization states with phase-inversion waves

Phase-inversion waves which are observed by these initial values are shown in Fig. 3, when $\alpha = 0.01$ and $\varepsilon = 0.05$. In the Fig. 3, each circle shows an attractor of each oscillator, vertical and horizontal axis are voltage and current, respectively. Each rectangle graph between the attractors is shown sum of voltages of adjacent oscillators along time. In other word, the graph expresses phase state between the adjacent oscillators along time. In-phase synchronization is expressed when large amplitude is shown, and anti-phase synchronization is observed when amplitude is almost zero. For example, a circle of corner of upper left shows an attractor of $OSC_{0,0}$, and next right box shows itinerancy of phase state between $OSC_{0,0}$ and $OSC_{0,1}$. In the Fig. 3, when all oscillators synchronize as an in-phase, we can observe that a phase-inversion wave which changes from anti-phase synchronization to in-phase synchronization propagates immediately after a phase-inversion wave which changes from in-phase synchronization to anti-phase synchronization propagates. Furthermore, we can observe that phase-inversion waves propagate in all columns and all rows at same time. In this paper, we call a set of these two phase-inversion waves "thin phase-inversion waves."

< IER of a phase state with thin phase-inversion waves>

We use initial-value-set A and initial-value-set B of which can be observed the phase state of Fig. 3. Itinerancy of $IER_{1,1}$ is shown in Fig. 4 in transient state. We can understand that value of the $IER_{1,1}$ is always under 1, and oscillation of the envelope curve is attenuating and is be-

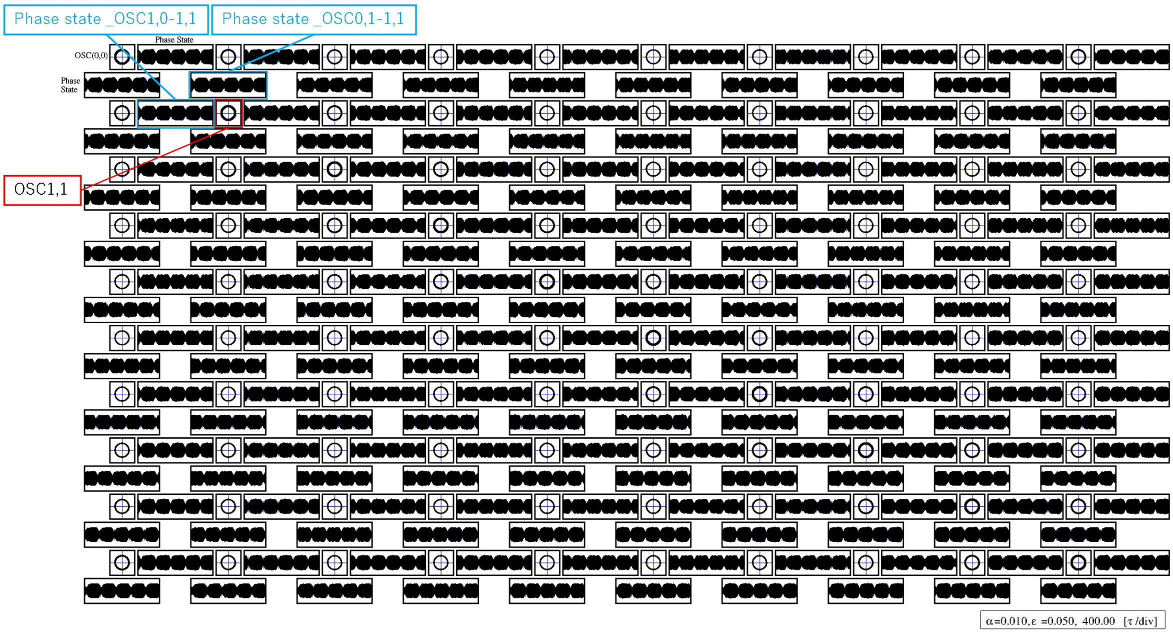

Figure 3: Synchronization state with propagating thin phase-inversion waves in each column and each row.

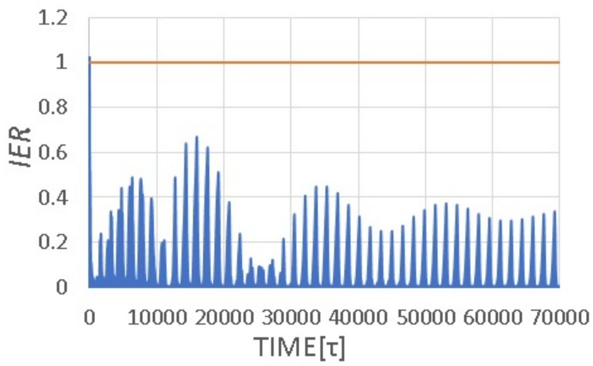

Figure 4: Behavior of $IER_{1,1}$ in transient state on the phase state of Fig. 3.

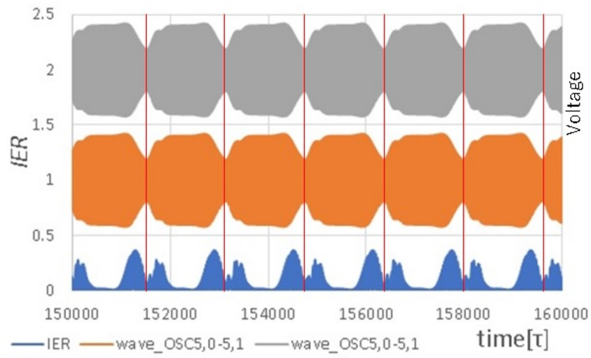

Figure 5: Comaprison among itinerancy of $IER_{1,1}$, itinerancies of a phase states between $OSC_{1,0}$ and $OSC_{1,1}$, and $OSC_{0,1}$ and $OSC_{1,1}$ on the phase state of Fig. 3.

coming a steady state. In other words, we can guess that the phase state is steady state. In Fig. 5, the itinerancy of $IER_{1,1}$ is compared with the itinerancies of phase states between $OSC_{1,0}$ and $OSC_{1,1}$, and $OSC_{0,1}$ and $OSC_{1,1}$ after disappearing the oscillation of the envelope curve. We can see that the $IER_{1,1}$ become large when thin phase-inversion waves arrive at $OSC_{1,1}$, and the $IER_{1,1}$ become small when thin phase-inversion waves pass the $OSC_{1,1}$.

< *IER* of a phase state with phase-inversion waves for vertical direction>

In this section, we investigate influences of only one thin phase-inversion wave for *IER*, because details of behaviors of *IER* by the phase-inversion waves are investigated.

We observe behavior of *IER* in when two thin phase-inversion waves are propagating in each column, and the phase state is in-phase synchronization without phase-inversion waves in each row (see Fig. 6). In each column, two thin phase-inversion waves propagate, collide each other at $OSC_{9,j}$ and $OSC_{0,j}$, reflect, and propagates to the opposite direction , respectively. The two thin phase-inversion waves propagate, arrive at $OSC_{4,j}$ and $OSC_{5,j}$, collide and reflect again. The two thin phase-inversion waves are continuously existing as above mentioned. An initial-value-set A and an initial-value-set B of which can be observed the phase state of Fig. 6 are used. Values of only $OSC_{1,1}$ of the initial-value-set B slightly differ from the initial-value-set A.

An itinerancy of $IER_{1,1}$ and an itinerancy of a phase state between $OSC_{0,1}$ and $OSC_{1,1}$ is shown in Fig. 7, $IER_{1,1}$ becomes large when a phase-inversion wave changing from the in-phase synchronization to the anti-phase synchronization is arrived at $OSC_{1,1}$, and $IER_{1,1}$ changes toward 0 after the phase-inversion wave passes $OSC_{1,1}$. $IER_{1,1}$ becomes large again when a phase-inversion wave changing from the

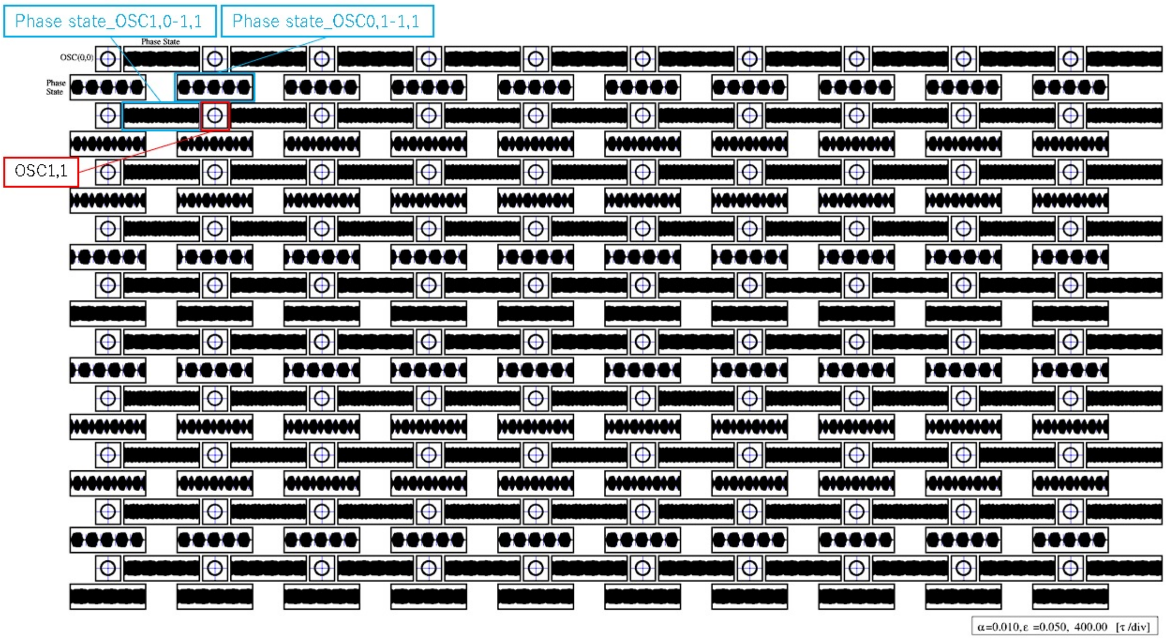

Figure 6: Synchronization state with propagating thin phase-inversion waves in each column.

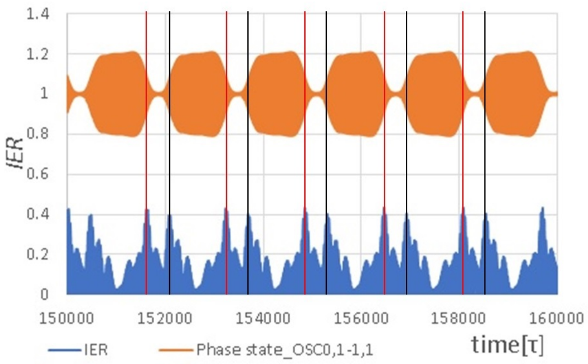

Figure 7: Comaprison between itinerancies of $IER_{1,1}$ and a phase states between $\mathrm{OSC}_{0,1}$ and $\mathrm{OSC}_{1,1}$ on the phase state of Fig. 6.

anti-phase synchronization to the in-phase synchronization is arrived at $\mathrm{OSC}_{1,1}$, and $IER_{1,1}$ changes toward 0 after the phase-inversion wave passes $\mathrm{OSC}_{1,1}$, again.

4. Conclusion

In this study, we observed synchronization phenomena on the torus of coupled 10×10 van der Pol oscillators. We observed that the phase-inversion waves were observed in each column and in each row on the torus. Furthermore, it was investigated that distance between two solution trajectories of instantaneous electric powers with two slightly different initial values, when the phase-inversion waves were propagating on the torus. We confirmed that $IER_{i,j}$ increased and decreased by which a phase-inversion wave passed at $\mathrm{OSC}_{i,j}$. In other words, distance between solution trajectories did not continuously increase, and decrease to almost 0 again. Therefore, we guess that these synchronization phenomena with propagating phase-inversion waves are stable.

Acknowledgements

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