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Nonlinear Control by Coupled Oscillator: from Nonholonomic Systems to Quasi-Passive Dynamic Walker

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Abstract—In this paper, we propose a novel control method for nonholonomic constrained systems using coupled oscillators composed of the Kuramoto model, which is one of mathematical models used to describe a synchronization phenomenon. The reason why we focus on the Kuramoto model is that it has the ability to generate stable rhythmic signals and it is sufficiently simple for mathematical analysis. We derive the stability criteria of Brockett Integrator with the proposed method and examine the effectiveness of the proposed method by several numerical simulations and experiments. Finally we apply the proposed method to control the walking distance of Quadrupedal Quasi-passive dynamic walker.

1. Introduction

In the field of nonlinear control theory, nonholonomic constrained systems have been intensively investigated since early 90's[1]. Nonholonomic system is a peculiar class of nonlinear systems which contains nonintegrable mechanical constraints. It can be controllable in nonlinear sense by considering the effect of Lie brackets of the input vector-fields although it is uncontrollable in linear sense. Such a system often appears in locomotion control problem of mobile robots or space robots.

As one of control strategy of nonholonomic system, it was shown that periodic inputs with proper phase gap such as a sine-cosine pair were effective. Here, the problem is how to create periodic inputs with a proper phase gap. In our previous work[2], Matsuoka model, which is a mathematical model of neural elements proposed by Matsuoka[3], was used to generate such periodic control inputs. The proposed control strategy was applied to rolling sphere control problem and its effect was verified with some numerical simulations. However, mathematical stability analysis of the proposed method has yet been conducted enough.

In this paper, we propose a novel control method for nonholonomic constrained systems based on the Kuramoto model, which is one of the mathematical models used to describe a synchronization phenomenon. The reason why the Kuramoto model is used is that it has the ability to generate stable rhythmic signals and it is sufficiently simple for mathematical analysis. Then, we derive the stability criteria with the proposed method and examine the effectiveness of the proposed method by several numerical simulations with Brockett integrator and two-wheeled vehicle. Finally, we apply the proposed method to control of quadrupedal quasi-passive dynamic walking robot.

2. Brockett integrator

At first, we focus on the so-called Brockett integrator system in particular, which is the simplest case of nonholonomic systems. The state equation of Brockett integrator is shown as follows.

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ X_2 & -X_1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \tag{1}$$

where $\boldsymbol{X} = (X_1, X_2, X_3)^T \in \mathbb{R}^3$ is the system state and $\boldsymbol{U} = (U_1, U_2)^T \in \mathbb{R}^2$ is the control input. It is clear that X_1 and X_2 can be controlled directly by U_1, U_2, while X_3 should be controlled indirectly by appropriate coordination of U_1 and U_2. For instance, it was shown that periodic inputs U_1 and U_2 with proper phase gap such as sine-cosine pair of functions[1] were effective. According to Stokes' theorem, Lie bracket of a pair of vector-fields corresponds to the displacement in the state space as a result of periodic input with sufficiently small amplitude.

Therefore, it can be expected that X_3 is able to be controlled with some periodic inputs U_1 and U_2. The problem is how to create periodic inputs with a proper phase gap. In this paper, we use well-known Kuramoto model to produce such periodic inputs.

3. Feedback control for nonholonomic systems using the Kuramoto model

3.1. Control of Brockett integrator

3.1.1. Kuramoto model

It is well known that there are many kinds of synchronization phenomena in autonomous system, such as mechanical system, chemical reaction and circadian rhythm. The Kuramoto model, first proposed by Kuramoto, is one of the mathematical models used to describe such synchronization[4]. The Kuramoto model consists of a population of N coupled phase oscillators θ_i having natural frequencies ω_i. For example, if $N = 2$, Kuramoto model can be denoted by

$$\frac{d\theta_1}{dt} = \omega_1 + k_{12}\sin(\theta_2 - \theta_1), \quad \frac{d\theta_2}{dt} = \omega_1 + k_{12}\sin(\theta_1 - \theta_2), \tag{2}$$

where k_{12} denotes the coupling strength between oscillators. This model is simple enough to be mathematically tractable, yet sufficiently complex to be nontrivial. As a result, it have been shown that this model can display a large variety of synchronization patterns.

3.1.2. *Proposed control method*

Based on the Kuramoto model (2), let us consider the following oscillator model with a feedback term.

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} (1 + K_p X_3)\omega_0 + k\sin(\theta_2 - \theta_1) \\ (1 - K_p X_3)\omega_0 + k\sin(\theta_1 - \theta_2) \end{pmatrix}, \tag{3}$$

where ω_0 is the common natural frequency, k is the bonding strength of oscillators and K_p is the feedback gain.

Using these oscillators θ_1 and θ_2, we propose to assign the following U_1 and U_2 as the control inputs for Brockett integrator (1),

$$U_1 = A\cos(\theta_1)\dot{\theta}_1, \quad U_2 = B\cos(\theta_2)\dot{\theta}_2. \tag{4}$$

3.1.3. *Stability Analysis*

In this section, we show that the proposed control inputs (4) can deliver X_3 in the system (1) to 0a if the phase difference between the oscillators (3) is sufficiently small, that is, $\sin(\theta_1 - \theta_2) \approx \theta_1 - \theta_2$.

Let Θ and $\dot{\Theta}$ be $\Theta := \theta_1 - \theta_2$ and $\dot{\Theta} := d\Theta/dt$. Brockett integrator system (1) with inputs (4) can be rewritten to

$$\begin{aligned} \dot{X}_3 &= -AB\omega_0 \sin\Theta + AB\sin(2\omega_0 t)(K_p\omega_0 X_3 - k\sin\Theta) \\ \dot{\Theta} &= 2(K_p\omega_0 X_3 - k\sin\Theta). \end{aligned}$$

When $\sin\Theta \approx \Theta$, these equations can be linearized to

$$\dot{X}_3 = -AB\omega_0\Theta + AB\sin(2\omega_0 t)(K_p\omega_0 X_3 - k\Theta) \tag{5}$$

$$\dot{\Theta} = 2(K_p\omega_0 X_3 - k\Theta). \tag{6}$$

The stability condition of the linearized system (5) and (6) is given by the following theorem.

Theorem 1. *The origin* $(X_3, \Theta) = (0, 0)$ *of (5), (6) is asymptotically stable if* $k > ABK_p\omega_0/2$.

Proof. At first, we introduce Krasovski theorem[5], which gives a sufficient condition for a stability of periodic dynamical system.

Krasovski Theorem. *Let* $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ *be a differential equation which is periodic in* t *with period* T*. Suppose there exists a* $\mathbb{C}^1$ *function* $V : I \times \Omega \to \mathbb{R}$*, periodic in* t *with period* T*, such that for some function* $a \in \mathbb{R}$ *and every* $(t, x) \in I \times \Omega$*:*

1) $V(t, \boldsymbol{x}) \geq a(\|\boldsymbol{x}\|)$; $V(t, 0) = 0$;

2) $\dot{V}(t, \boldsymbol{x}) \leq 0$;

3) *expect for the origin, M contains no complete positive semi-trajectory where* $M = \{(t, \boldsymbol{x}) \in I \times \Omega : \dot{V} = 0\}$;

are satisfied.

Then, the origin is uniformly asymptotically stable.

Using this theorem, we show the stability of the system (5), (6) .

Let $\boldsymbol{x}$ be $\boldsymbol{x} = (X_3, \Theta)$. We defined $V(t, \boldsymbol{x})$ and $a(\|\boldsymbol{x}\|)$ as follows,

$$V(t, \boldsymbol{x}) = 2(K_p\omega_0 X_3 - k\Theta)^2 + ABK_p\omega_0^2\Theta^2 \tag{7}$$

$$=: \alpha X_3^2 + 2\beta X_3\Theta + \gamma\Theta^2 \tag{8}$$

$$a(\|\boldsymbol{x}\|) = \frac{\alpha + \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2}(X_3^2 + \Theta^2). \tag{9}$$

It is obvious that $V(t, \boldsymbol{0}) = 0$. From Eq. (8) and (9), $V(t, \boldsymbol{x}) - a(\|\boldsymbol{x}\|)$ yields

$$\begin{aligned} &V(t, \boldsymbol{x}) - a(\|\boldsymbol{x}\|) \\ &= \frac{\alpha - \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2}X_3^2 + 2\beta X_3\Theta + \frac{-\alpha + \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2}\Theta^2 \\ &= \left(\sqrt{\frac{\alpha - \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2}}X_3 + \sqrt{\frac{-\alpha + \gamma + \sqrt{(\alpha - \gamma)^2 + 4\beta^2}}{2}}\Theta\right)^2 \\ &\geq 0. \end{aligned} \tag{10}$$

From the inequality, it is verified that the condition 1) in Krasovski Theorem is satisfied.

Next, differentiating $V(t, \boldsymbol{x})$ with respect to t, $\dot{V}(t, \boldsymbol{x})$ can be derived as

$$\begin{aligned} \dot{V} &= 4(K_p\omega_0 X_3 - k\Theta)(K_p\omega_0\dot{X}_3 - k\dot{\Theta}) + 2ABK_p\omega_0^2\dot{\Theta}\Theta \\ &= 2\dot{\Theta}\{K_p\omega_0(-AB\omega_0\Theta + AB\sin(2\omega_0 t)\dot{\Theta}/2) - k\dot{\Theta}\} \\ &\quad + 2ABK_p\omega_0^2\dot{\Theta}\Theta \\ &= (ABK_p\omega_0\sin(2\omega_0 t) - 2k)\dot{\Theta}^2 \qquad (11) \\ &\leq (ABK_p\omega_0 - 2k)\dot{\Theta}^2. \qquad (12) \end{aligned}$$

From Eq. (12), it can be said that the condition 2) in Krasovski Theorem is satisfied if $k > ABK_p\omega_0/2$.

Finally, from Eq. (11), (6), M can be described as:

$$M = \{(t, X_3, \Theta) : K_p\omega_0 X_3 - k\Theta = 0\} \tag{13}$$

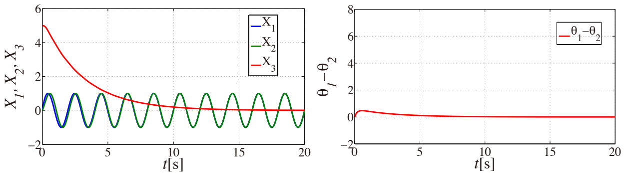
Suppose a $(t^*, X_3^*, \Theta^*) \in M$. If a trajectory which starts from (t^*, X_3^*, Θ^*) are always contained in M, the equation $K_p\omega_0\dot{X}_3 - k\dot{\Theta} = 0$ has to be satisfied at least. From Eq. (5), (6), the time evolution of (X_3, Θ) from (t^*, X_3^*, Θ^*) can be derived as:

$$\dot{X}_3 = \omega_0\Theta^*, \quad \dot{\Theta} = 0. \tag{14}$$

Therefore, only the origin can stay on M, and then, it is verified that the condition 3) in Krasovski Theorem is satisfied.

The above discussion provides that all conditions of Krasovski Theorem are satisfied if $k > ABK_p\omega_0/2$. Therefore, the origin $\boldsymbol{x} = (X_3, \Theta) = (0, 0)$ of the system (5), (6) is asymptotically stable if $k > ABK_p\omega_0/2$. □

From the theorem, it can be said that the proposed control method can deliver X_3 to 0 if the phase difference between the oscillators is sufficiently small.



(a) X_1, X_2, X_3 (b) Phase difference $\theta_1 - \theta_2$

Figure 1: Control of Brockett integrator with proposed method

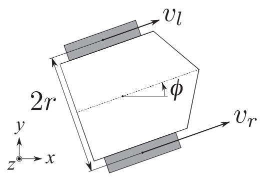


Figure 2: Kinematic model of two-wheeled robot

3.1.4. Simulation Result

We verified the proposed control method by some numerical simulations. Figure 1 shows the simulation results when the initial state was $\boldsymbol{X} = [5, 0, 0]^T$ and the feedback gain was $K_p = 0.1$ in Eq. (3). Other parameters were set as $k = \pi$, $\omega_0 = \pi$ and $A = B = 1$ such that the condition $k > ABK_p\omega_0/2$ is satisfied.

Figure 1(a) shows the trajectory of X_1, X_2 and X_3 and Fig. 1(b) shows the phase difference $\theta_1 - \theta_2$. It can be seen that X_3 successfully converged to 0 and the phase difference $\theta_1 - \theta_2$ also converged to 0. Therefore, it was verified that X_3 in Brockett integrator (1), which corresponds to the direction of Lie bracket, can be controlled by the proposed control method (3) .

3.2. Control of two-wheeled vehicle

In this section, we apply the proposed approach to control of two-wheeled vehicle, which is one of the nonholonomic systems as well as Brockett integrator.

3.2.1. Transformation from Brockett integrator to Two-wheeled vehicle

Figure 2 shows the kinematic model of two-wheeled vehicle. Under non-slip and non-slide assumptions, the kinematic equation of the two-wheeled vehicle model is shown as follows,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos\phi/2 & \cos\phi/2 \\ \sin\phi/2 & \sin\phi/2 \\ 1/2r & -1/2r \end{pmatrix} \begin{pmatrix} v_r \\ v_l \end{pmatrix}, \qquad (15)$$

where v_r and v_l are velocities of right and left wheel respectively, θ is the attitude of the vehicle and $2r$ is the vehicle width.

In non-linear control theory, it is known that the two-wheeled vehicle system is classified into the same class of nonholonomic system as the Brockett integrator and these two systems are equivalent under coordinates and inputs transformations. Such transformation between Brockett integrator (1) and the two-wheeled robot model (15) can be derived as following,

$$X_1 = x, \; X_2 = \tan\phi, \; X_3 = 2y - x\tan\phi \qquad (16)$$

and

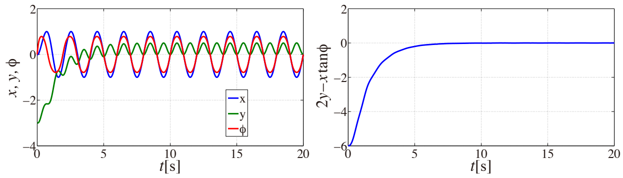
$$v_r = \frac{U_1}{\cos(\tan^{-1} X_2)} + r\cos^2(\tan^{-1} X_2)U_2 \qquad (17)$$

$$v_l = \frac{U_1}{\cos(\tan^{-1} X_2)} - r\cos^2(\tan^{-1} X_2)U_2. \qquad (18)$$

Under the non-slip and non-slide constraints, the two-wheeled vehicle can not move toward the positive direction of y-axis, which is vertical to the wheels in a moment. Then, in order to control y of the two-wheeled vehicle (15), we try to control X_3 in Eq. (16) with the proposed control method (3) and (4).

3.2.2. Simulation Result

We confirm an effect of proposed method with numerical simulation. The physical parameter is set as $2r = 0.24$. The control parameters in Eq. (3) are set as $k = \pi$, $\omega_0 = \pi$, $K_p = 0.2$ and $A = B = 1$. Figure 3 shows the simulation result with the initial state $(x, y, \phi)^T = (0, 0, -3)^T$. Figure 3(a) and 3(b) show the trajectory of (x, y, ϕ) and $X_3 = 2y - x\tan\phi$ respectively. Although y continued to oscillate a little, the trajectory of y successfully converged to near 0 and $X_3 = 2y - x\tan\phi$ converged to 0. From this result, it is verified the proposed control method (3) can be also applied to the control of y in the two-wheeled vehicle, which corresponds to the vertical direction to the wheels.



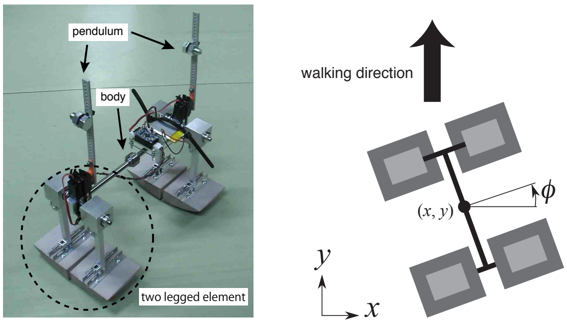
(a) (x, y, ϕ) (b) $X_3 = 2y - x\tan\phi$

Figure 3: Control of two-wheeled vehicle.

4. Feedback Control of Quadrupedal Quasi-passive Dynamic Walking Robot

4.1. Quadrupedal quasi-passive dynamic walking Robot "Duke"

Finally, we apply the proposed control method to control of quadrupedal quasi-passive dynamic walking robot. Passive Dynamic walking(PDW) refers to a class of mechanical devices that are able to walk down a shallow incline without any actuation and can realize walking by very



(a) Overview of “Duke” (b) coordinate of (x, y, ϕ)

Figure 4: Quasi-passive dynamic walking robot “Duke”

simple mechanisms[6]. Therefore, much attention has been paid to PDW as an approach to realize walking robot[7, 8].

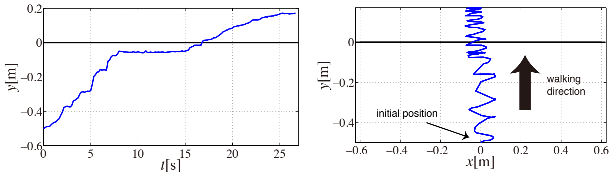
In our previous research[9], we developed Quasi-passive dynamic walking robot “Duke”(Fig. 4). This robot was designed based on PDW robot and was able to walk on the level ground only by exerting measly energy only in the lateral direction using invert pendulums of each two legged element(See Fig. 4(a)). In addition, as a result of analyses based on the nonlinear control theory, we revealed that Duke had a nonholonomic constraint that was corresponds to that of two-wheeled vehicle. And we concluded that the locomotion of Duke consisted of Lie bracket motion derived from a sphere rolling motion of the sole and a leg swinging motion. Therefore, we apply the proposed control method (4) to the control of the walking distance of Duke.

4.2. Experimental Result

We confirm an effect of proposed method through walking experiments with the quadrupedal quasi-PDW robot Duke. Let us define the posture angle ϕ and position (x, y), which is the center of the robot, as shown in Fig. 4(b). To associate with the result in Section 3.2, the walking direction of Duke was coincided with the positive direction of a y-axis.

Figure 5 shows the experiment result when the initial state was $(x(0), y(0), \phi(0))^T = (0, -0.5, 0)^T$. The parameters were set as $k = 3\pi$, $\omega_0 = 2\pi/1.5$, $K_p = 2$, $A = 0.1$ and $B = 0.3$. Figure 5(a), 5(b) show the walking distance y and the trajectory (x, y) respectively. At first, Duke walked forward while repeating lateral oscillation of each two-legged element. Gradually Duke approached $y^* = 0$ and the phase difference of lateral oscillation and the walking velocity became small. Finally, Duke became to keep stamping around $y \approx 0.15$[m].

From the result, the effectiveness of the proposed control method for the control of Duke was confirmed to some extent. However, some steady-state deviation was observed. This is thought to be due to a leg swinging motion, which was not considered in the two-wheeled vehicle model. To reduce the error, it may be effective to integrate a integrator



(a) walking distance y (b) Trajectory of (x, y)

Figure 5: Experimental: feedback control of walking distance

into the controller. This issue is left as a future work.

5. Conclusion

In this paper, we proposed the feedback control method for nonholonomic constrained systems based on the Kuramoto model and showed the stability of Brockett integrator system with the proposed control method. Through numerical simulations, we verified that the control method can control Brockett integrator and two-wheeled vehicle, which are well-known nonholonomic constrained system. Finally we applied the proposed method to control the walking distance of Quadrupedal quasi-PDW robot Duke.

The stability analysis in this paper was limited by the assumption that the phase difference is relatively small. To apply it to more complicated systems, it should be extended in the future work.

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