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Multipopulation Replicator Dynamics with Capitation Tax and Subsidy

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Abstract—In population consists of many selfish players, the purpose of each player often conflicts with the total purpose of the population. In order to govern the population, the "government" imposes a capitation tax and offers a subsidy. To describe dynamical evolutions of a population state which is a distribution of strategies in such a population, replicator dynamics with a capitation tax and a subsidy has been proposed. The model deals with interactions of players in a single population. However, some social or biological systems consists of multiple populations. In such a situation, games are played between two players who belong to not only the same population but different populations. In this paper, we extend the model to describe changes of multiple populations' states.

1. Introduction

In evolutionary games, the purpose of each player often conflicts with the total purpose of the population because of his/her selfish behavior. In order to govern the population, the "government" collects payoffs as capitation taxes and reallocates them as subsidies. Kanazawa *et al.* has proposed replicator dynamics with a capitation tax and a subsidy to model such a situation [1]. In this model, the government is willing to lead the population to a desirable target state. However, some social or biological systems consist of multiple populations. In such a situation, games are played between two players who belong to not only the same population but different populations. So, we have to deal with interactions between players in multiple populations.

In this paper, we propose multipopulation replicator dynamics with a capitation tax and a subsidy. To control all populations to a desirable target state, the government imposes a capitation tax and offers a subsidy depending on the target states to players in each population. Moreover, we discuss conditions of the capitation tax and the subsidy which make the target state asymptotically stable in our model.

2. Multipopulation Replicator Dynamics with Capitation Tax and Subsidy

Several concepts of multipopulation evolutionary games have been proposed [2, 3, 4]. In this paper, we employ a model proposed by Taylor [3].

Suppose that $I = \{1, \ldots, n\}$ is a set of populations and P^i is the number of players who belong to population i. Let $S^i = \{1, \ldots, n^i\}$ $(i \in I)$ be a set of pure strategies of the population i and $N = \sum_{i \in I} n^i$. Suppose that $x^i_k \in R$ $(i \in I, k \in S^i, 0 \leq x^i_k \leq 1)$ is the proportion of players with a strategy k in the population i. $x^i = (x^i_1, x^i_2, \ldots, x^i_{n^i})^T$ is a population state which describes a distribution of strategies in the population i. A population state combination of all populations is denoted by $x = (x^{1^T}, x^{2^T}, \ldots, x^{n^T})^T$, we call it population state of all populations for simplicity. In addition, let $\Delta^i \subset R^{n^i}$ and $\Delta = \times_{i \in I} \Delta^i$ be spaces of population states of the population i and all populations, respectively. Denoted by $\mathrm{bd}(\Delta)$ and $\mathrm{int}(\Delta)$ are the boundary and the interior of Δ, respectively. $C(x^i)(:= \{i \in S^i | x^i_k > 0\})$ is an index set of all nonzero elements of x^i. Assume that a player's payoff depends on the current population state of all populations. In this paper, we define a payoff function of a player with the strategy k in the population i by $u^i(e^i_k, x) = e^i_k \Sigma_{j \in I} A^{ij} x^j (x \in \Delta)$, where e^i_k is the n^i-dimensional unit vector such that the kth element equals 1 and $A^{ij} \in R^{n^i \times n^j}$ is the payoff matrix of the population i against the population j. An average payoff of the population i is given by $u^i(x^i, x)$. A state $x^* \in \Delta$ is called a Nash equilibrium if $u^i(x^{i*}, x^*) \geq u^i(x^i, x^*)$ holds for all $x^i \in \Delta^i$ and for all $i \in I$. Replicator dynamics which describes evolutions of population states is given by

$$\dot{x}^i_k = \{u^i(e^i_k, x) - u^i(x^i, x)\} x^i_k. \tag{1}$$

This equation shows that the number of players with strategy k increases when they earn larger payoffs than the average payoff in the population i.

In our model, the government imposes a capitation tax t^i on each player and offers a subsidy $c^i P^i$ to population i. Let $x^* = (x^{1*^T}, x^{2*^T}, \ldots, x^{n*^T})^T$ be a target state of the government, where $x^{i*} = (x^{i*}_1, \ldots, x^{i*}_{n^i})^T$. Let $P^i_k > 0$ be the number of players who adopt strategy k in population i. We assume that a subsidy for player with strategy k in population i depends on P^i_k. The offered subsidy $c^i P^i x^{i*}_k$ is equally-divided to players with strategy k. Then, each player earns the following subsidy:

$$\frac{c^i P^i x^{i*}_k}{P^i_k} = c^i \frac{x^{i*}_k}{x^i_k}. \tag{2}$$

Thus, the payoff function with the capitation tax and the subsidy is given by

$$\tilde{u}^i(e^i_k, x) = u^i(e^i_k, x) - t^i + c^i \frac{x^{i*}_k}{x^i_k}, \tag{3}$$

and the average payoff of population i is given by

$$\tilde{u}^i(x^i, x) = u^i(x^i, x) - t^i + c^i. \tag{4}$$

Then we obtain multipopulation replicator dynamics with the capitation tax and the subsidy as follows:

$$\dot{x}^i_k = \{u^i(e^i_k, x) - u^i(x^i, x)\}x^i_k + c^i(x^{i*}_k - x^i_k). \tag{5}$$

Eq. (5) can be interpreted as a controlled system with the state feedback which is given by $c^i(x^{i*}_k - x^i_k)$. So, c^i can be considered as a feedback gain.

3. Stability Conditions of the Target State

In this section, we discuss stability conditions of the target state in Eq. (5).

Proposition 1 (Invariance Under a Local Shift) *Equation (5) is invariant under any local shift of the payoff matrix A^{ij} for all $i, j \in I$.*

Note that the local shift of A^{ij} is the addition of a constant to all elements of a column of A^{ij}. This proposition allows us to assume that each element of A^{ij} is non-negative for all $i, j \in I$ without loss of generality.

Proposition 2 (Equilibrium Target State) *If the target state is an equilibrium point of Eq.* (1)*, then it is always an equilibrium point of Eq.* (5).

According to Proposition 2, if we adopt an equilibrium point of Eq. (1) as a target state, we have only to discuss stability of the target state in Eq. (5). Therefore, we focus on the case that the target state is an equilibrium point of Eq. (1) .

For simplicity, we begin with the case that $c = c^1 = c^2 = \cdots = c^n$ holds. In this case, we have the following theorem for locally asymptotic stability of the target state.

Theorem 1 (Locally Asymptotic Stability with $c = c^i$) *Let the linearization system of Eq. (1) at the target state $x = x^*$ be $\dot{x} = J_0 x$. Then, the linearization system of Eq. (5) at $x = x^*$ is given by*

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \vdots \\ \dot{x}^n \end{bmatrix} = \left(J_0 + \begin{bmatrix} cI_{n^1} & 0 & \dots & 0 \\ 0 & cI_{n^2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & \dots & cI_{n^n} \end{bmatrix} \right) x, \tag{6}$$

where I_{n^i} is the $n^i \times n^i$ unit matrix. Let the eigenvalues of the Jacobian matrix J_0 be λ_{0k} ($k = 1, \dots, N$). The origin is asymptotically stable in Eq. (6) *if and only if $c > \max_k(\Re(\lambda_{0k}))$ holds, where $\Re(\lambda_{0k})$ is the real part of λ_{0k}.*

For the case that the subsidy c^i of the population $i \in I$ is determined independently of the other populations, we have the following theorem:

Theorem 2 (Locally Asymptotic Stability) *The linearization system of Eq. (5) at $x = x^*$ is given by*

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \vdots \\ \dot{x}^n \end{bmatrix} = \left(J_0 + \begin{bmatrix} c^1 I_{n^1} & 0 & \dots & 0 \\ 0 & c^2 I_{n^2} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & \dots & c^n I_{n^n} \end{bmatrix} \right) x. \tag{7}$$

Let $J_{0(l,m)}$ be the element of the lth row and the mth column of J_0, and $d(i) = \sum_{0 \le j \le i} n^j$. The origin is asymptotically stable in Eq. (7) *if the following inequality holds for all $i \in I$:*

$$c^i > \max_{k \in S^i} \left(\sum_{r \neq d(i)+k} |J_{0(d(i)+k,r)}| + J_{0(d(i)+k,d(i)+k)} \right). \tag{8}$$

If the origin of the linearization system Eq. (6) is asymptotically stable, then the target population state $x = x^*$ of the nonlinear system Eq. (5) is locally asymptotically stable.

Theorem 3 (Globally Asymptotic Stability) *Suppose that each element of A^{ij} is non-negative for all $i, j \in I$ and the target state $x = x^*$ is a Nash equilibrium. The target state is a globally asymptotically stable equilibrium point of Eq.* (5)*, if the following equation holds: for all $i \in I$*

$$c^i > \max\{0, \sup_{x \in \mathrm{int}(\Delta) \setminus x^*} \bar{c}(x)\}, \tag{9}$$

where the function $\bar{c}(x)$ is defined by

$$\bar{c}(x) := -\frac{\sum_{i \in I} \sum_{k \in S^i} (x^{i*}_k - x^i_k) u^i(e^i_k, x)}{\sum_{i \in I} \sum_{k \in C(x^{i*})} (x^{i*}_k - x^i_k) \frac{x^{i*}_k}{x^i_k}} \tag{10}$$

for all $x \in \mathrm{int}(\Delta) \setminus x^*$. Moreover, $\bar{c}(x)$ satisfies

$$\bar{c}(x) \le 2\bar{a} \sum_{i \in I} (n^i - 1), \tag{11}$$

where $\bar{a} = \max_{i,j \in I} \max_{k \in S^i, l \in S^j} A^{ij}_{kl}$ and A^{ij}_{kl} is the element of the kth row and the lth column of payoff matrix A^{ij}.

By Proposition 1, without the loss of generality, it can be assumed that each element of payoff matrices is non-negative. For a game with payoff matrices which have negative elements, we can obtain an another game whose payoff matrices have no negative elements by a local shift to the original game. Applying Theorem 3 to the local shifted game, we obtain the subsidy which makes the target state of the original game globally asymptotically stable.

In Theorem 3, we assume that the target state is a Nash equilibrium. Any equilibrium point which is not a Nash equilibrium must be on the boundary of Δ, that is, $x^* \in \mathrm{bd}(\Delta)$. For such a target state, we have the following theorem:

Theorem 4 (Convergence Condition to the Boundary) *Suppose that each element of A^{ij} is non-negative for all $i, j \in I$ and the target state x^* satisfies $x^* \in \mathrm{bd}(\Delta)$. If the inequality*

$$c^i > \max_{m \notin C(x^{i*})} \max_{l \in I} \max_{n \in S^l} A^{il}_{mn} \tag{12}$$

holds for all $i \in I$, then $\lim_{t\to\infty} x^i_k(t) = 0$ for all $k \notin C(x^{i*})$.

If $x^{i*}_k = 0$ holds, then $\{x \in \Delta | x^i_k = 0\} \subset \mathrm{bd}(\Delta)$ is a positive invariant set since $\dot{x}^i_k = 0$ also holds. By Theorem 4, for all $i \in I$, we calculate a subsidy c^i_1 for which $x^i_k(t)$ converges to 0 for all strategies $k \notin C(x^{i*})$. Then, by Theorem 3, for a sub-game which eliminates strategies $k \notin C(x^{i*})$, we calculate a subsidy c^i_2 which makes the target state globally asymptotically stable. Finally, we set the subsidy $c^i = \max\{c^i_1, c^i_2\}$ so that the target state becomes an attractor whose basin is Δ. However, it is not obvious whether the subsidy obtained by the above procedure can make the target state an attractor.

When the target state is on the vertex, we can prove that it is not only an attractor but a globally asymptotically stable equilibrium point.

Corollary 1 (Stability of the Target State on a Vertex) *Suppose that it x^{i*} is on a vertex of Δ^i for all $i \in I$. If the inequality $c^i > n \max_{k \in S^i \subset m} \max_{l \in I} \max_{t \in S^l} a^{il}_{kt}$ holds for all $i \in I$, then the target state is globally asymptotically stable in Eq. (5).*

4. Two-Population Two-Strategy Game

In this section, we focus on a two-population two-strategy game and investigate locally asymptotic stability conditions for the target state.

We set the payoff matrices A^{11}, A^{12}, A^{21}, and A^{22} as follows:

$$A^{11} = A^{12} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \tag{13}$$

$$A^{21} = A^{22} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}. \tag{14}$$

Suppose that population states of populations 1 and 2 are (x^1_1, x^1_2) and (x^2_1, x^2_2), respectively. Note that $x^1_1 + x^1_2 = 1$ and $x^2_1 + x^2_2 = 1$ hold. For simplicity, we denote a population state of all populations by (x, y), where $x = x^1_1 = 1 - x^1_2$ and $y = x^2_1 = 1 - x^2_2$. When we offer subsidies c^1 and c^2 to populations 1 and 2, respectively, replicator dynamics is given by

$$\dot{x} = x(1-x)\{(a_1+a_2)(x+y) - 2a_2\} + c^1\{x^* - x\}, \tag{15}$$

$$\dot{y} = y(1-y)\{(b_1+b_2)(x+y) - 2b_2\} + c^2\{y^* - y\}. \tag{16}$$

Since we assume the target state (x^*, y^*) is an equilibrium point of Eq. (1), it satisfies the following equations:

$$x^*(1-x^*)\{(a_1+a_2)(x^*+y^*) - 2a_2\} = 0, \tag{17}$$

$$y^*(1-y^*)\{(b_1+b_2)(x^*+y^*) - 2b_2\} = 0. \tag{18}$$

Table 1: Values of G and H for each target state.

	$(0,0)$	$(\tilde{x},0)$	$(\tilde{x},1)$	$(1,0)$	$(1,1)$
G	$-2a_2$	$\frac{2a_2(a_1-a_2)}{a_1+a_2}$	$\frac{2a_1(a_2-a_1)}{a_1+a_2}$	a_1+a_2	$-2a_2$
H	$-2b_2$	$\frac{2(a_2b_1-a_1b_2)}{a_1+a_2}$	$\frac{2(a_2b_1-a_1b_2)}{a_1+a_2}$	b_1-b_2	$-2b_2$

Equation (17) (*resp.* Eq. (18)) holds only if $x^* = 0$, $1 - x^* = 0$, or $(a_1+a_2)(x^*+y^*) - 2a_2 = 0$ (*resp.* $y^* = 0$, $1 - y^* = 0$, or $(b_1+b_2)(x^*+y^*) - 2b_2$) holds. Obviously, $x = 0, \tilde{x}, 1$ and $y = 0, \tilde{y}, 1$ satisfy Eqs. (17) and (18), where $\tilde{x} = 2a_2/(a_1+a_2) - y^*$ and $\tilde{y} = 2b_2/(b_1+b_2) - x^*$. So, there are 9 candidates for the target state (x^*, y^*). However, we have the same game by swapping the populations 1 and 2, and the pure strategies 1 and 2. So, we deal with the following 6 target states:

$$(x^*, y^*) = (0,0), (\tilde{x},0), (\tilde{x},\tilde{y}), (\tilde{x},1), (1,0), (1,1). \tag{19}$$

Note that $(x^*, y^*) = (\tilde{x}, \tilde{y})$ exists in Δ if and only if $a_2 = \alpha a_1$ and $b_2 = \alpha b_1$ hold for a constant α. Then, $(\tilde{x}, \tilde{y})$ satisfies $\tilde{x} + \tilde{y} = 2a_2/(a_1+a_2) = 2b_2/(b_1+b_2) = 2\alpha/(1+\alpha)$.

We denote Eqs. (17) and (18) by $\dot{x} = f_x(x,y)$ and $\dot{y} = f_y(x,y)$, respectively. The Jacobian matrix J of Eqs. (17) and (18) at the target state is given by

$$J = \begin{bmatrix} \left.\frac{\partial f_x}{\partial x}\right|_{x^*,y^*} & \left.\frac{\partial f_x}{\partial y}\right|_{x^*,y^*} \\ \left.\frac{\partial f_y}{\partial x}\right|_{x^*,y^*} & \left.\frac{\partial f_y}{\partial y}\right|_{x^*,y^*} \end{bmatrix}, \tag{20}$$

where

$$\frac{\partial f_x}{\partial x} = (1-2x)\{(a_1+a_2)(x+y) - 2a_2\} + x(1-x)(a_1+a_2) - c^1, \tag{21}$$

$$\frac{\partial f_x}{\partial y} = x(1-x)(a_1+a_2), \tag{22}$$

$$\frac{\partial f_y}{\partial x} = y(1-y)(b_1+b_2), \tag{23}$$

$$\frac{\partial f_y}{\partial y} = (1-2y)\{(b_1+b_2)(x+y) - 2b_2\} + y(1-y)(b_1+b_2) - c^2. \tag{24}$$

We begin with the case that the target state (x^*, y^*) is not $(\tilde{x}, \tilde{y})$. In this case, the Jacobian matrix has at least one non-diagonal zero element since the target state satisfies $y^* = 0$ or $y^* = 1$. So, two eigenvalues of J correspond to diagonal elements of J.

If we define G and H by

$$G = (1-2x^*)\{(a_1+a_2)(x^*+y^*) - 2a_2\} + x^*(1-x^*)(a_1+a_2), \tag{25}$$

$$H = (1-2y^*)\{(b_1+b_2)(x^*+y^*) - 2b_2\} + y^*(1-y^*)(b_1+b_2), \tag{26}$$

then we obtain $G < c^1$ and $H < c^2$ as locally asymptotically stabilization conditions of the target state. Table 1 shows the corresponding values of G and H for each target state.

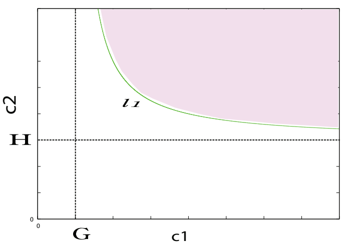

Figure 1: Stability region for $G > 0, H > 0$.

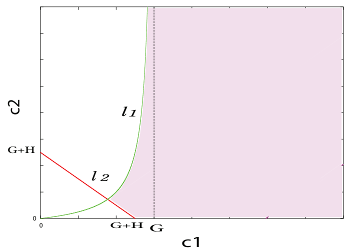

Figure 2: Stability region for $G > 0, H < 0$.

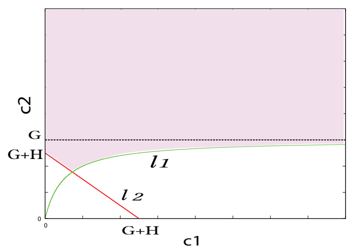

Figure 3: Stability region for $G < 0, H > 0$.

In the case that the target state (x^*, y^*) is $(\tilde{x}, \tilde{y})$, we have $G = \tilde{x}(1 - \tilde{x})(a_1 + a_2)$ and $H = \tilde{y}(1 - \tilde{y})(b_1 + b_2)$. The Jacobian matrix is given by

$$J = \begin{bmatrix} G - c^1 & G \\ H & H - c^2 \end{bmatrix}. \quad (27)$$

Its characteristic equation is given by

$$\lambda^2 + (c^1 + c^2 - G - H)\lambda - Gc^2 - Hc^1 + c^1c^2 = 0. \quad (28)$$

Both of the real parts of the eigenvalues of J are negative if and only if the following conditions hold:

$$-Gc^2 - Hc^1 + c^1c^2 > 0, \quad (29)$$

$$c^1 + c^2 - G - H > 0. \quad (30)$$

Figures 1, 2, and 3 show the stabilization regions of the target state in the c^1-c^2 plane. Note that the target state is locally asymptotically stable in Eq. (5) without subsidies if both G and H are negative. So, we exclude cases that both of G and H are negative. The curves and the lines in Figs. 1 and 2 are

$$\begin{cases} l_1 : c^2 = \frac{H}{1 - \frac{G}{c^1}}, \\ l_2 : c^2 = -c^1 + G + H. \end{cases}$$

Dotted lines in Figs. 1, 2, and 3 are asymptotes of a curve l_2.

In the white region of Figs. 1, 2, and 3, the target state is not stable equilibrium point. The boundary l_1 between colored and white regions is a pitchfork bifurcation set, where two stable equilibrium points and the unstable target state collide. On the other hand, the boundary l_2 is a Hopf bifurcation set, where a stable limit cycle and the unstable target state collide. Thus, the target state is asymptotically stabilized in the colored region.

Table 2 shows the stabilization conditions of the target state obtained by Theorems 1 and 2 depending on G and H. The stabilization conditions of the target state obtained by eigenvalues of the Jacobian matrix include the stabilization conditions given by Theorems 1 and 2 as shown in Table 2, which implies that the stabilization conditions given by Theorems 1 and 2 are unnecessarily conservative in our model. It is our future work to show less conservative stabilization conditions than that provided in this paper.

Table 2: The stabilization conditions obtained by Theorems 1 (the condition of c) and 2 (the condition of (c_1, c_2)).

	(x^*, y^*)	$G > 0, H > 0$	$G > 0, H < 0$	$G < 0, H > 0$
Th. 1	$(0,0), (1,0), (1,1)$	$\max(G, H)$	G	H
	$(\tilde{x}, 0), (\tilde{x}, 1)$	$\max(G, H)$	G	H
	$(\tilde{x}, \tilde{y})$	$G + H$	$\max(G + H, 0)$	$\max(G + H, 0)$
Th. 2	$(0,0), (1,0), (1,1)$	(G, H)	$(G, 0)$	$(0, H)$
	$(\tilde{x}, 0), (\tilde{x}, 1)$	$(2G, H)$	$(2G, 0)$	$(0, H)$
	$(\tilde{x}, \tilde{y})$	$(2G, 2H)$	$(2G, 0)$	$(0, 2H)$

5. Conclusions

In this paper, we have proposed multipopulation replicator dynamics with a capitation tax and a subsidy. We have discussed the conditions of the subsidy which make the target state locally or globally asymptotically stable. Using a two-population two-strategy game, we have investigated the regions of subsidies in which the target state is locally asymptotically stable. It is our future work to show less conservative stabilization conditions than that provided in this paper.

Acknowledgments

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