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Edge Consensus on Complex Networks: A Structural Analysis

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Abstract—Differing from the existed works centralize the nodal consensus, this paper shifts its focus to the consensus taking place on the edges (i.e., edge consensus) of complex networks and investigates the influence of network structure on its edge consensus. By transforming the original nodal topology to its corresponding line graph, we analyze the influence of network structure on some performances based on simulations, including the edge consensus convergence speed, edge degree distribution, and edge assortative property.

1. Introduction

Consensus is a typical collective behavior which aims to guide all agents of a group to reach a common state using only local information. And a large number of efforts have been devoted to this interesting research field [1, 2, 3, 4, 5]. Recently, with the rapid development of network science, the problem of consensus has been extended to complex networks and becomes one of the main focuses in network research [6, 7, 8, 9, 10, 11, 12, 13]. References [9, 10] devoted to the consensus of small-world networks, while [11, 12, 13] turned to that of scale-free networks.

However, all the existed letters, as the references stated above, kept their eyes on the nodal dynamics and investigated the consensus of complex networks in which consensus means all the nodes can reach an agreement. Very recently, Vicsek *et al.* paid attention to the edges of complex networks in [14]. They gave a specific evolution dynamics of the edges and compared the controllability properties of the edge dynamics with those of nodal dynamics. Furthermore, in [15], the authors built an effective edge consensus protocol for the first time and solved the edge consensus of complex networks.

Motivated by these two works, in this paper, we still concern the edge consensus, but switch our attention to the influence of the network structure by taking the scale-free networks which follow the power law degree distribution with a scaling exponent γ between 2 and ∞ as examples. For scale-free networks with the same number of nodes and edges, apparently, different degree distribution means different structure. This paper analyzes the influence of network structure on edge consensus convergence speed, edge degree distribution, edge assortative property by changing the degree distribution of scale-free network, i.e., by changing the scaling exponent γ of its degree distribution. Based on simulation results, we can get that the edge consensus convergence speed increases fast as the scaling exponent γ increases. Also, when the original network is with scaling exponent $\gamma \in (3, +\infty)$, the edge degree distribution has an approximate power-law tail. Moreover, it is found that the edge assortative coefficient is a decreasing function of the assortative coefficient of its original network, when its original network is heterogeneous with the scaling exponent $\gamma \in (2, 3)$.

2. Preliminaries

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote an undirected graph with node set $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ and edge set $\mathcal{E} = \{(v_i, v_j)|$ if there exists an edge between v_i and $v_j\}$. The communication topology of the network is described by the adjacent matrix $A = \left(a_{ij}\right) \in R^{N\times N}$, where

$$a_{ij} = a_{ji} = \begin{cases} = & 1, \text{if } (v_i, v_j) \in \mathcal{E}; \\ = & 0, \text{otherwise}. \end{cases} \qquad (1)$$

On the basis of the nodal topology adjacent matrix A, at first, we will present how to get the corresponding edge topology from the original nodal topology by virtue of line graph [17, 18]. Before moving on, it is worthy to know that two edges are neighboring edges if they have a common ending node [15]. For an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the evolution of an undirected graph to its line graph can be divided into two steps: (*i*) we focus on the edge set $\mathcal{E}$ and transform the all the edges $(v_i, v_j) \in \mathcal{E}$ into nodes denoted as ij and named as freshly generated nodes; (*ii*) add an edge to the two freshly generated nodes if their corresponding original edges in $\mathcal{E}$ are neighboring edges. The detailed description can consult reference [15].

The edge consensus protocol is [15]:

$$\begin{aligned} x_{ij}(k+1) &= x_{ij}(k) + \varepsilon\Bigg[\textstyle\sum_{s\in N(i)} a_{is}\left[x_{is}(k) - x_{ij}(k)\right] \\ &+ \textstyle\sum_{s\in N(j)} a_{js}\left[x_{js}(k) - x_{ij}(k)\right]\Bigg], \forall (v_i, v_j) \in E, \end{aligned} \qquad (2)$$

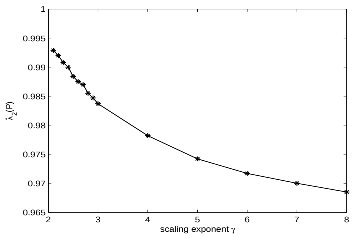

Figure 1: Second largest eigenvalue of P as a function of γ.

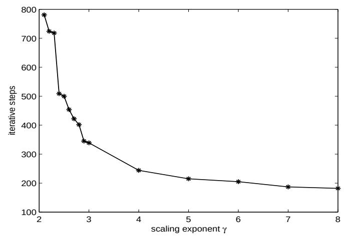

Figure 2: Edge consensus convergence speed of the SF network as a function of γ.

where x_{ij} represents the dynamics of the edge (v_i, v_j), $\varepsilon > 0$ is the step size and $N(i) = \{j|(v_i, v_j) \in E\}$ is the neighboring set of node i.

As stated in reference [15], if the step size $\varepsilon > 0$ satisfies

$$0 < \varepsilon < \Delta = \frac{1}{\max\{\sum_{s\in N(i),s\neq j} a_{is} + \sum_{s\in N(j),s\neq i} a_{js}\}}, \qquad (3)$$

and the original nodal topology is connected, protocol (2) can guarantee the edge consensus. We can prettify Eq. (2) to the matrix form as

$$X(k + 1) = PX(k), \qquad (4)$$

and P is a double stochastic matrix. Define the initial value of X as $X(0)$, one has

$$X(k) = P^k X(0). \qquad (5)$$

By [16], the convergence speed of (4) is determined by the second largest eigenvalue of P, $\lambda_2(P)$. Moreover, the smaller the $\lambda_2(P)$ is, the faster the consensus converges.

In what follows, we will study some performance of the edges of scale-free networks from the perspective of structure.

3. Main results

To explore the influence of network structure, we consider an undirected and unweighted scale-free (SF) network which follows a power law degree distribution with a tunable scaling exponent $\gamma \in (2, \infty)$ [19, 20, 21].

In the simulations, the number of nodes is N, the initial edge number is m_0, the edge number connecting from the new node to the old nodes is m.

3.1. Influence of network structure on edge consensus convergence speed

This subsection presents the edge consensus convergence speed of scale-free networks with different degree distribution. We take $N = 500$, $m_0 = 3$, $m = 3$ and the simulation results are averaged over 50 realizations.

Figure 1 shows that the second largest eigenvalue of P, $\lambda_2(P)$, is a decreasing function of the scaling exponent γ, which means that the more homogeneous the original network, the faster the convergence speed of edge consensus. Figure 2 shows that the time steps required to achieve consensus also decreases with the increasing scaling exponent γ and this result is in accordance with the change trend of $\lambda_2(P)$. Here the consensus is achieved if $\max_{(v_i,v_j)\in\mathcal{E}} - \min_{(v_i,v_j)\in\mathcal{E}} \leq 0.001$. These two figures illustrate that the edge consensus convergence speed is the increasing function of the scaling exponent γ of the original nodal network.

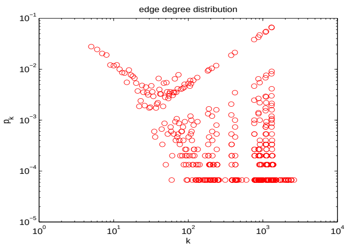

Figure 3: The edge degree distribution when its original network with scaling exponent $\gamma = 2.1$.

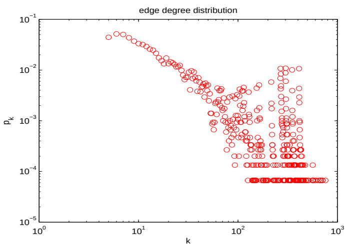

Figure 4: The edge degree distribution when its original network with scaling exponent $\gamma = 2.5$.

3.2. Influence of network structure on edge degree distribution

At first, it should be stated that the edge degree distribution of a network is the degree distribution of its corresponding line graph. Under this statement, some simulations are presented to display the edge degree distributions of scale-free networks. Here $N = 5000$, $m_0 = 3$ and $m = 3$.

Figures 3–7 show the edge degree distributions of scale-free networks with different scaling exponent γ. From all these figures, we can gain that:

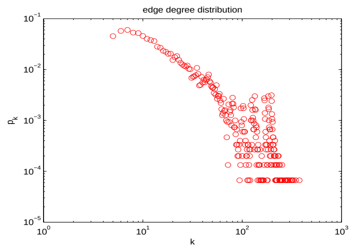

Figure 5: The edge degree distribution when its original network with scaling exponent $\gamma = 3$.

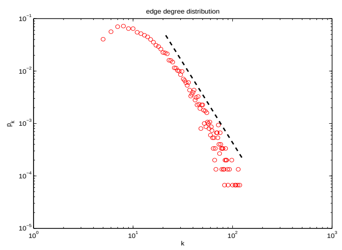

Figure 6: The edge degree distribution when its original network with scaling exponent $\gamma = 6$. The dashed line have slope $\gamma_{edge} = 2.48$.

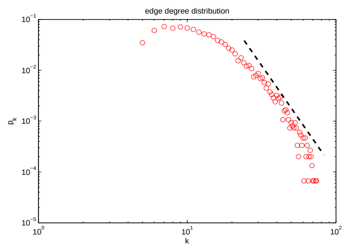

Figure 7: The edge degree distribution when its original network with scaling exponent $\gamma = 10$. The dashed line have slope $\gamma_{edge} = 2.82$.

- (i) for $\gamma \in (2, 3)$, as shown in figures 3, 4, 5, the edge degree distribution does not emerge distinguished feature;

- (ii) for $\gamma \in (3, \infty)$, the edge degree distribution has an approximate power-law tail, as shown in figures 6, 7.

In a word, the edge degree distribution of a scale-free network is affected by the structure of its original network. For a heterogeneous original network with scaling exponent $\gamma \in (2, 3)$, the edge degree distribution has no obvious feature; whereas for an approximative homogeneous original network, that is the original network with scaling exponent $\gamma > 3$, the edge degree distribution has an approximate power-law tail.

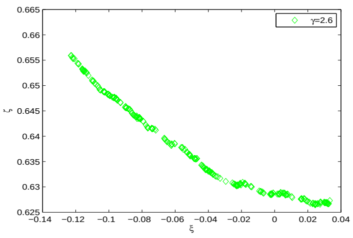

Figure 8: The relationship between ξ and ζ when its original network with scaling exponent $\gamma = 2.6$.

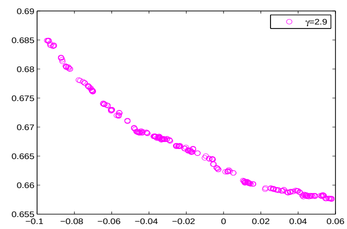

Figure 9: The relationship between ξ and ζ when its original network with scaling exponent $\gamma = 2.9$.

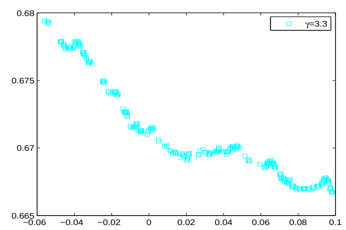

Figure 10: The relationship between ξ and ζ when its original network with scaling exponent $\gamma = 3.3$.

3.3. Influence of network structure on edge assortative property

The edge assortative property of a network means the assortative property of the line graph of its original network. For the network has degree correlation, assortative coefficient or disassortative coefficient is another important measure of complex network to further depict its degree correlation. The objective of this subsection is to demonstrate the relationship between the assortative property of the original scale-free network ξ, and its edge assortative property ζ.

In this subsection, $N = 500$, $m_0 = 3$, $m = 3$. Figures 8,9,10 show that the edge assortative coefficient ζ is a roughly decreasing function of that of its original scale-free network ξ, when the network is heterogeneous with $\gamma \in (2, 3)$. Here, we can not obtain the specific lower bound of the scaling exponent γ but only get this property from the simulations. What's more, the edge assortative coefficients of scale-free networks are always positive, that is the network corresponds to the line graph is always assortative, no matter its original network is assortative or not.

4. Conclusions

This paper explores the influence of network structure on edge consensus by taking the scale-free networks with different degree distribution as examples, and finds out the relation of the edge consensus convergence speed and the scaling exponent γ of the degree distribution. Similarly, the influence of network structure on edge degree distribution and edge assortative property are discussed and some interesting results are obtained.

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