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The Multiple-Scales Method for Nonlinear Circuits with Very Small and Very Large Natural Frequencies

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Abstract—In this paper, we propose a multiple-scales method for finding an approximate solution of state equations describing nonlinear circuits that contain both very large and very small natural frequencies. The state equations are represented by the system of autonomous nonlinear differential equations. The proposed multiple-scales method is based on a matrix function and a perturbation technique. As an example, we use the Rayleigh's equation and show that the envelope of an oscillator can be calculated separately.

1. Introduction

Nonlinear circuits are key building blocks in many electric and electronic systems. Because of their significance, we need to devise a precise analysis of an appropriate circuit model. In the mathematical model, stray and sluggish elements cannot be neglected. The former affects the high-frequency behaviour (fast-dynamics) and the latter the low-frequency behaviour (slow-dynamics) of the circuits. Hence, the circuit behaviour is characterised by a mixture of low-frequency and high-frequency behaviour [5]. We call such systems stiff nonlinear circuits.

When we analyse stiff nonlinear circuits, we should consider both the fast- and slow-dynamics or, in other words, the dynamics on the multiple time scales. The most common approach is a straightforward numerical method such as Runge-Kutta. However, widely separated time constants cause long simulation times. Moreover, we may also encounter numerical instability during simulations.

Stiff linear systems are described in [1, 6, 7]. The multiple-scales methods are mainly used for the analysis of mechanical systems described by the linear ordinary differential equations of 2nd order [4]. However, in electric and electronic circuits, more general description by the state vector and matrix notation is needed because the number of dynamic elements is larger. This paper presents the multiple-scales method for the stiff nonlinear circuits by using a matrix function and a perturbation technique.

2. Fundamental equation

Let us consider nonlinear circuits with no sources. Their mathematical models can be represented by an autonomous system of differential equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = A\boldsymbol{x} + \varepsilon\{B\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x})\}, \tag{1}$$

where A and B are $n \times n$ real matrices and $\boldsymbol{x}$ is n real vector called a state vector. The vector valued function $\boldsymbol{f}(\boldsymbol{x})$ is the smooth nonlinear function. We assume that the spectrum $\lambda(A)$ has two scales, which implies that the eigenvalues λ_i $(i = 1, \cdots, n)$ are clustered into two sets widely separated in magnitude. Here, we call (1) a stiff nonlinear system.

3. Multiples-scales method

3.1. Preconditioning

Let the eigenvalues of A be denoted by ρ_i and $\alpha_i + \mathrm{j}\,\beta_i$, where ρ_i is a real number and $\mathrm{j} = \sqrt{-1}$. If α_i and ρ_i are small, the amplitudes of oscillations slowly decay or diverge away. If β_i is small, the amplitude oscillates slowly. In these cases, these amplitudes remain almost constant over the fast time scale. The slow and fast time scales provide the division of the eigenvalues into two groups:

1. $\alpha_i, \quad \beta_i, \quad \rho_i$ are $O(1)$
2. $\alpha_k, \quad \beta_k, \quad \rho_k (i \neq k)$ are $O(\varepsilon)$.

In the latter case, let

$$\alpha_k = \varepsilon\alpha'_k, \quad \beta_k = \varepsilon\beta'_k, \quad \rho_k = \varepsilon\rho'_k, \tag{2}$$

where α'_k, β'_k, and $\rho'_k = O(1)$. Following the above division of the eigenvalues, we have

$$\begin{aligned} P^{-1}AP &= D_0 + \varepsilon D_1 \\ D_0 &= \mathrm{diag}(\cdots, \alpha_i + \mathrm{j}\,\beta_i, \cdots, \rho_i, \cdots) \\ D_1 &= \mathrm{diag}(\cdots, \alpha'_k + \mathrm{j}\,\beta'_k, \cdots, \rho'_k, \cdots), \end{aligned} \tag{3}$$

where P is $n \times n$ nonsingular matrix. Hence, we divide A as

$$A = A_0 + \varepsilon B_1, \tag{4}$$

where

$$A_0 = PD_0P^{-1}, \; B_1 = PD_1P^{-1}. \tag{5}$$

If $\alpha_i = 0$ and $\rho_i = 0$, the matrix A_0 is called oscillatory. When $A_0 = A$ and $B_1 = B$, we obtain (1). We deal with the case where A is oscillatory.

3.2. First approximate solution

We introduce various time scales defined by

$$\tau_0 = t, \quad \tau_1 = \varepsilon t, \quad \tau_2 = \varepsilon^2 t, \ \cdots . \tag{6}$$

The time scale τ_0 is the fast scale, and $\tau_2, \tau_3, \cdots$ are slow scales. We are now in a position to obtain the asymptotic solution of (1) in terms of $\tau_0, \tau_1, \tau_2, \cdots$. Hence, we should transform the independent variable t into $\tau_0, \tau_1, \tau_2, \cdots$. Let us define the differential operators $D = \dfrac{\mathrm{d}}{\mathrm{d}t}$, $D_0 = \dfrac{\partial}{\partial \tau_0}$, $D_1 = \dfrac{\partial}{\partial \tau_1}$, $D_2 = \dfrac{\partial}{\partial \tau_2}, \cdots$. Then, (6) gives the operator transformation by using the chain rule

$$D = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots . \tag{7}$$

Let the asymptotic solution of (1) be of the form

$$\boldsymbol{x} = \boldsymbol{x}^{(0)}(\tau_0, \tau_1, \tau_2, \cdots) + \varepsilon\, \boldsymbol{x}^{(1)}(\tau_0, \tau_1, \tau_2, \cdots) + \cdots . \tag{8}$$

Substituting (8) into (1) with the use of (7) and equating the same powers of ε, we obtain the series of differential equations

$$(D_0 - A)\boldsymbol{x}^{(0)} = \boldsymbol{0} \tag{9-a}$$
$$(D_0 - A)\boldsymbol{x}^{(1)} = -(D_1 - B)\boldsymbol{x}^{(0)} + \boldsymbol{f}(\boldsymbol{x}^{(0)}) \tag{9-b}$$
$$(D_0 - A)\boldsymbol{x}^{(2)} = -D_2\, \boldsymbol{x}^{(0)} - (D_1 - B - \boldsymbol{f}'(\boldsymbol{x}^{(0)}))\boldsymbol{x}^{(1)} \quad \cdots\cdots\cdots \tag{9-c}$$

where $\boldsymbol{f}'(\boldsymbol{x}^{(0)})$ is the Jacobian matrix of $\boldsymbol{f}(\boldsymbol{x})$ at $\boldsymbol{x} = \boldsymbol{x}^{(0)}$. The general solution of (9-a) is given by

$$\boldsymbol{x}^{(0)} = \exp(A\tau_0)\, \boldsymbol{y}(\tau_1, \tau_2, \cdots), \tag{10}$$

where $\boldsymbol{y}(\tau_1, \tau_2, \cdots)$ is n vector determined by $\tau_1, \tau_2, \cdots$ in higher approximations. Substituting (10) into the right hand side of (9-b), we obtain

$$D_0\, \boldsymbol{x}^{(1)} - A\, \boldsymbol{x}^{(1)} = -\exp(A\tau_0)[\{D_1 - B^{(1)}(\tau_0)\}\, \boldsymbol{y} - \boldsymbol{f}^{(1)}(\tau_0,\ \boldsymbol{y})], \tag{11}$$

where

$$B^{(1)}(\tau_0) = \exp(-A\tau_0) B \exp(A\tau_0)$$
$$\boldsymbol{f}^{(1)}(\tau_0,\ \boldsymbol{y}) = \exp(-A\tau_0)\boldsymbol{f}(\exp(A\tau_0)\, \boldsymbol{y}).$$

In order to remove the secular term from the solution $\boldsymbol{x}^{(1)}$ of (11), the vector $\boldsymbol{y}$ should be determined to eliminate the constant vector (dc component) on the right hand side of (11) [2, 3]. Hence, we obtain the constant vector

$$\boldsymbol{c}(\boldsymbol{y}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \{B^{(1)}(\xi)\boldsymbol{y} + \boldsymbol{f}^{(1)}(\xi,\ \boldsymbol{y})\}\mathrm{d}\xi \tag{12}$$

Using the vector $\boldsymbol{c}(\boldsymbol{y})$, we derive the condition for avoiding secular term in (11)

$$D_1\, \boldsymbol{y} - \boldsymbol{c}(\boldsymbol{y}) = \boldsymbol{0}. \tag{13}$$

Equation (13) is the system of nonlinear equations. Its solution can be written as

$$\boldsymbol{y}(\tau_1, \tau_2, \cdots) = Y(\tau_1)\boldsymbol{z}(\tau_2), \tag{14}$$

where $\boldsymbol{Y}(\tau_1)$ is $n \times n$ matrix and $\boldsymbol{z}(\tau_2)$ is n vector. Substituting (14) into (10), we obtain the solution

$$\boldsymbol{x}^{(0)}(\tau_0, \tau_1) = \exp(A\tau_0)Y(\tau_1)\boldsymbol{z}(\tau_2). \tag{15}$$

Hence, the solution to the first approximation is

$$\boldsymbol{x}^{(0)}(t) = \exp(At)Y(\varepsilon t)\boldsymbol{z}(0), \tag{16}$$

where $\boldsymbol{z}(0)$ is the initial value of $\boldsymbol{x}$ at $t = 0$.

3.3. Second approximate solution

We now proceed to the second approximation. In order to determine $\boldsymbol{z}(\tau_2)$ in (15), we set the condition that the particular solution $\boldsymbol{x}^{(2)}$ of (9-c) has no secular term. Now, we consider (14) and introduce a new vector $\boldsymbol{c}'$ defined as

$$\boldsymbol{c}'(\boldsymbol{z}, \tau_0, \tau_1, \tau_2) = B^{(1)}(\tau_0)\, \boldsymbol{y}(\tau_1, \tau_2) + \boldsymbol{f}^{(1)}(\tau_0, \boldsymbol{y}) - \boldsymbol{c}(\boldsymbol{y}). \tag{17}$$

If (13) is satisfied, the particular solution of (11) is given by

$$\boldsymbol{x}^{(1)} = -\exp(A\tau_0) \int_0^{\tau_0} \boldsymbol{c}'(\boldsymbol{z}, \xi, \tau_1, \tau_2)\mathrm{d}\xi. \tag{18}$$

Substituting (18) and (15) into (9-c), we obtain

$$D_0\, \boldsymbol{x}^{(2)} - A\, \boldsymbol{x}^{(2)} = -\exp(A\tau_0)\{Y(\tau_1)D_2\, \boldsymbol{z}(\tau_2) + (-D_1 + B^{(1)}(\tau_0) + \boldsymbol{f}'^{(1)}(\tau_0))\, \boldsymbol{c}'^{(1)}\} \tag{19}$$

where

$$\boldsymbol{c}'^{(1)}(\boldsymbol{z}, \tau_0, \tau_1, \tau_2) = \int_0^{\tau_0} \boldsymbol{c}'(\boldsymbol{z}, \xi, \tau_1, \tau_2)\mathrm{d}\xi$$
$$\boldsymbol{f}'^{(1)}(\tau_0) = \exp(-A\tau_0)\boldsymbol{f}'(\exp(A\tau_0)Y_0\boldsymbol{z})\exp(A\tau_0).$$

Taking the dc-component of the right handside of (19) with respect to τ_0 gives

$$\boldsymbol{d}(\boldsymbol{z}) = \lim_{T\to\infty} \int_0^T \{(-D_1 + B^{(1)}(\xi) + \boldsymbol{f}'^{(1)}(\xi))\, \boldsymbol{c}'^{(1)}(\boldsymbol{z}, \xi, \tau_1, \tau_2)\}\mathrm{d}\xi. \tag{20}$$

We then obtain the condition for the absence of the secular term in (19)

$$D_2\, \boldsymbol{z} + Y^{-1}(\tau_1)\boldsymbol{d}(\ \boldsymbol{z}) = 0. \tag{21}$$

The solution of (21) can be expressed as

$$\boldsymbol{z} = \boldsymbol{z}(\tau_2) = Z(\tau_2)\, \boldsymbol{z}_0, \tag{22}$$

where $Z(\tau_2)$ is the matrix introduced to represent the solution $\boldsymbol{z}(\tau_2)$ and $\boldsymbol{z}_0$ is a constant vector. Substituting (21) into (18) and (15), we obtain

$$\boldsymbol{x}^{(0)} = \exp(A\tau_0)Y(\tau_1)Z(\tau_2)\, \boldsymbol{z}_0 \tag{23}$$

and

$$\boldsymbol{x}^{(1)} = -\exp(A\tau_0)\ \boldsymbol{c}'^{(1)}(Z(\tau_2), \tau_0, \tau_1, \tau_2). \qquad (24)$$

Therefore, the solution to the second approximation is

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{x}^{(0)}(t, \varepsilon t, \varepsilon^2 t) + \varepsilon \boldsymbol{x}^{(1)}(t, \varepsilon t, \varepsilon^2 t) \\ &= \exp(At)\{Y(\varepsilon t)Z(\varepsilon^2 t) - \varepsilon\ \boldsymbol{c}'^{(1)}(Z(\varepsilon^2 t), t, \varepsilon t)\}.\end{aligned} \qquad (25)$$

In practice, the processes of obtaining the second approximate solution becomes rather involved.

4. Illustrative example

For simplicity, we consider the two-dimensional nonlinear oscillatory system and apply the proposed method. Let us consider Rayleigh's equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + u - \varepsilon\left(\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{1}{3}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^3\right) = 0. \qquad (26)$$

Equation (26) is reduced to (1) by setting $v = \mathrm{d}u/\mathrm{d}t$, where

$$\boldsymbol{x} = \begin{pmatrix} v \\ u \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} \frac{1}{3}v^3 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that A is oscillatory. The first approximate solution takes the form

$$\begin{aligned}&\boldsymbol{x}(\tau_0, \tau_1)^{(0)} = \exp(A\tau_0)\boldsymbol{y}(\tau_1) \\ &= \begin{pmatrix} \cos\tau_0 & -\sin\tau_0 \\ \sin\tau_0 & \cos\tau_0 \end{pmatrix}\begin{pmatrix} y_1(\tau_1) \\ y_2(\tau_1) \end{pmatrix}.\end{aligned} \qquad (27)$$

The vector $\boldsymbol{y}(\tau_1)$ is determined under the condition of no secular term (13) and is given by the solution of the nonlinear system of equations

$$\left.\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}\tau_1} - \frac{1}{2}y_1 + \frac{1}{8}(y_1^3 + y_1 y_2^2) = 0 \\ \frac{\mathrm{d}y_2}{\mathrm{d}\tau_1} - \frac{1}{2}y_2 + \frac{1}{8}(y_2^3 + y_1^2 y_2) = 0\end{aligned}\right\}. \qquad (28)$$

Both solutions y_1 and y_2 of (28) are nonzero. As the system approaches the steady-state, the solutions become $y_1 \to 0$ and $y_2 \to \pm 2$ or $y_2 \to 0$ and $y_1 \to \pm 2$. In order to find the approximate solutions of (28), we assume that the solution $\boldsymbol{y}$ is given by

$$\left.\begin{aligned}y_1(\tau_1) = a(\tau_1)\cos\omega(\tau_1) \\ y_2(\tau_1) = a(\tau_1)\sin\omega(\tau_1)\end{aligned}\right\}. \qquad (29)$$

Substituting (29) into (28) and equating the coefficients of $\cos(\omega(\tau_1))$ and $\sin(\omega(\tau_1))$, respectively, we obtain the nonlinear equation

$$\left.\begin{aligned}&\frac{\mathrm{d}a}{\mathrm{d}\tau_1} - \frac{1}{2}a + \frac{1}{8}a^3 = 0 \\ &\frac{\mathrm{d}\omega}{\mathrm{d}\tau_1} = 0\end{aligned}\right\}. \qquad (30)$$

Integrating both equations (30), we have

$$\left.\begin{aligned}a(\tau_1) &= \frac{2}{\sqrt{1 + \exp(-\tau_1 - c_1)}} \\ \omega(\tau_1) &= \omega_0\end{aligned}\right\}, \qquad (31)$$

where c_1 and ω_0 are the initial values of a and ω at $\tau_1 = 0$, respectively. Hence, we have

$$\left.\begin{aligned}y_1(\tau_1) &= \frac{2}{\sqrt{1 + \exp(-\tau_1 - c_1)}}\cos\omega_0 \\ y_2(\tau_1) &= \frac{2}{\sqrt{1 + \exp(-\tau_1 - c_1)}}\sin\omega_0.\end{aligned}\right\}. \qquad (32)$$

We then obtain

$$\left.\begin{aligned}v^{(0)}(\tau_0, \tau_1) &= \frac{2}{\sqrt{1 + \exp(-\tau_1 - c_1)}}\cos(\tau_0 + \omega_0) \\ u^{(0)}(\tau_0, \tau_1) &= \frac{2}{\sqrt{1 + \exp(-\tau_1 - c_1)}}\sin(\tau_0 + \omega_0)\end{aligned}\right\} \qquad (33)$$

Finally,

$$\left.\begin{aligned}y_1(\tau_1) &= \frac{2}{\sqrt{1 + \exp(-\tau_1 - c_1)}}\cos\omega_0 \\ y_2(\tau_1) &= \frac{2}{\sqrt{1 + \exp(-\tau_1 - c_1)}}\sin\omega_0\end{aligned}\right\}. \qquad (34)$$

As an example, let us set the initial condition of (26) to

$$u(0) = a_0, \quad v(0) = 0. \qquad (35)$$

Then, we have

$$\omega_0 = \frac{\pi}{2}, \quad \exp(-c_1) = \frac{4}{a_0^2} - 1. \qquad (36)$$

Substituting these values into (32) gives the first approximate solution

$$\left.\begin{aligned}v^{(0)}(t) &= \frac{2}{\sqrt{1 + (\frac{4}{a_0^2} - 1)\exp(-\varepsilon t)}}\sin t \\ u^{(0)}(t) &= \frac{2}{\sqrt{1 + (\frac{4}{a_0^2} - 1)\exp(-\varepsilon t)}}\cos t\end{aligned}\right\}. \qquad (37)$$

The three graphes shown in Fig.1 demonstrate the effectiveness of the multiple-scales method by decreasing the parameter ε. The solid lines show the first approximate solution with the multiple-scales method. The dotted lines are obtained with numerical integration available in the Octave tool. When decreasing the parameter ε from 0.6 to 0.2, both transient curves get closer. For $\varepsilon \leq 0.1$, there is no visible difference.

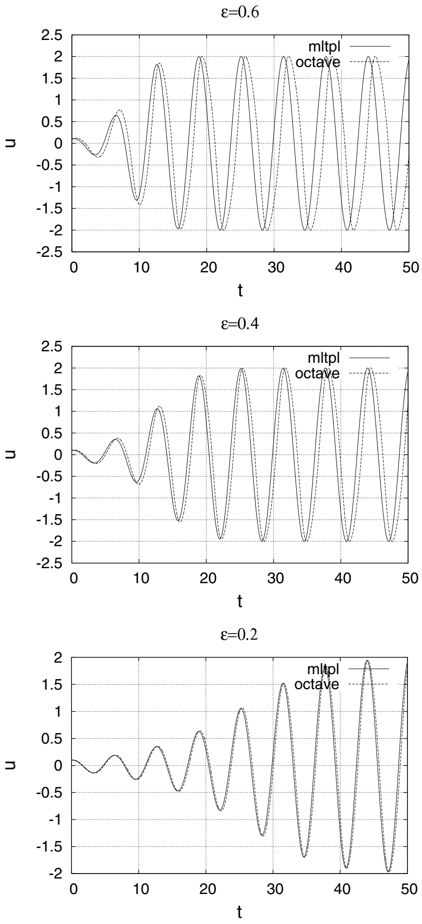

Figure 1: Comparison of the proposed method of multiple scales with numerical integration of Rayleigh's equation for $\varepsilon = 0.6,\ 0.4$, and 0.2. The notation 'eps' denotes ε.

The solution of (28) that determines the first approximation can be obtained by any numerical method. In Fig. 2, the solution $y_2(\tau_1)$ is shown for $\epsilon = 0.2$. The envelope of the solution can be computed separately. This is the merit of the multiple-scales method.

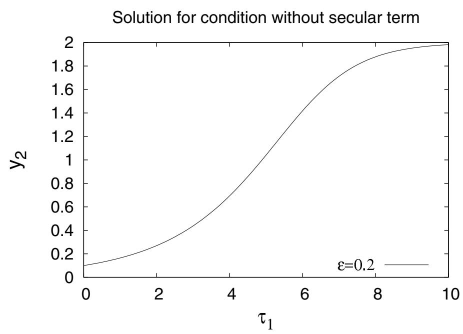

Figure 2: The slow-mode of the oscillation for $\epsilon = 0.2$.

5. Conclusion

We presented the multiple-scales method for the analysis of stiff nonlinear circuits expressed by autonomous system of differential equations. At each stage of higher approximations, we compute the solution of a set of nonlinear differential equations that are derived to avoid the presence of the secular term. For higher dimensional systems, the numerical computation of the product of the matrix exponential is essential to obtain an accurate solution at each step of approximation. In the given example, the solution was obtained analytically by hand caluculations.

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