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Convergence Analysis of Discrete-Time Multi-Agent Systems Based on Sequential Connectivity

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Abstract—It is well known that the jointly connectivity and sequential connectivity are two fundamental concepts in multi-agent systems. We discover that the jointly connectivity is equivalent to sequential connectivity if the communication topology has self-loops at each node. Based on the above result, the consensus of discrete-time multi-agent systems is then given under the condition of jointly connectivity by constructing a sequentially connected sequence. The above proof greatly simplified the former proofs.

1. Introduction

A multi-agent system (MAS) is a system consisted of multiple interacting intelligent agents. Examples include birds flocks, sheep herds, fish schools, online trading, disaster response, multi-robot coordination, economic systems, and so on. Over the past decades, numerous models are proposed to characterize and analyze MAS, including Vicsek model ([1]), Boid model ([2]), linear iteration model ([3], [4], [9], [7]), and so on. Recently, MAS has received an increasing attention from mathematics, physics, engineering sciences, and social communities ([6], [8], [10]).

As one of the simplest MAS models, linear iteration model updates the state of each agent by using the linear average of all states of its neighbors ([7], [11], [12]). Some known results have indicated that all states of the linear iteration model will converge to some identical value under the condition of jointly connectivity ([4], [7], [5], [9]) or sequential connectivity ([10]). As pointed out in [10], the sequential connectivity is a much more stronger condition than the jointly connectivity and the joint connectivity does not implies the sequential connectivity. In this paper, we will bridge the gap between the sequential connectivity and the jointly connectivity under some suitable conditions.

According to [4], [5], and [9], the linear MAS can converge exponentially under the condition of jointly connectivity. It should be pointed out that the above results are obtained based on the theory of infinite matrices products introduced by Wolfowitz in [3]. From [3], the convergence condition of the products of infinite matrices is that there exists a uniformly lower bound for the nonzero entries of all matrices. However, in this paper, we will prove that the above uniformly lower bound condition is not always necessary for the consensus of the linear MAS.

In [7], Blondel and his colleagues introduced a new approach to prove the consensus of discrete-time MAS without using the matrices products. Unfortunately, the above method cannot be generalized to the case of joint graph with a spanning tree. To overcome the uniformly lower bound condition of matrices products approach in [3] and the limitation of method in [7], a new technique is proposed to prove the consensus of discrete-time multi-agent systems under the same condition of jointly connectivity in this paper, called sequential connectivity approach.

This paper is organized as follows. Section 2 introduces several necessary preliminaries on graph theory and matrix theory. The problem is formulated in Section 3. In Section 4, a novel approach is then given to prove the consensus of discrete-time MAS. Finally, some concluding comments or suggestions are given in Section 5.

2. Preliminaries

A graph $\mathcal{G} = (V, E)$ is composed of two sets V and E, where $V = \{1, 2, ..., N\}$ is the set of nodes and $E \subseteq V \times V$ is the set of edges. There exists a path from node i to j if and only if there exist k different nodes $\{i_s\}$ with $1 \leq s \leq k$, $i_1 = i$, $i_k = j$ satisfying $(i_p, i_{p+1}) \in E$ for any $1 \leq p \leq s-1$. Denote $\langle i, j \rangle \in \mathcal{G}$ if there exists a path from node i to j.

In graph $\mathcal{G}$, if there exists a node i_0 which has paths to any other nodes, then the graph $\mathcal{G}$ contains a spanning tree with the root i_0. For the above graph $\mathcal{G}$, if there exist paths from any node $i \in V$ to any node $j (\neq i) \in V$, then the graph is called strongly connected.

For graph $\mathcal{G} = (V, E)$ and $S \subseteq V$. Define

$$\mathcal{N}(S, \mathcal{G}) = \{j \in V : \exists i \in S, (i, j) \in E\}$$

be the set of neighbors of S.

For different graphs $\mathcal{G}_k = (V, E_k)$ $(1 \leq k \leq K)$ with the same set of vertices V, the union of these K graphs is defined as

$$\bigcup_{k=1}^{K} \mathcal{G}_k = (V, \bigcup_{k=1}^{K} E_k).$$

A matrix is called nonnegative if each of its entry is nonnegative. A nonnegative matrix is called stochastic if the sum of each row equals to 1. A matrix is 0-1 if each entry is 0 or 1. For two matrices $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$, define $A \circ B = (\max\{a_{ij}, b_{ij}\})_{n\times n}$. At the same time, define $\prod_{i=1}^{K} A_i = A_K A_{K-1} \cdots A_1$ be the left products of matrices.

For each non-negative matrix $A = \{a_{ij}\}_{i,j=1}^{N}$, a graph $\mathcal{G}(A) = (V, E)$ with $E \subseteq V \times V$ is given to characterize the structure of A, where $V = \{1, 2, ..., N\}$ and $a_{ij} > 0$ iff $(j, i) \in E$.

Hereafter, two fundamental concepts are given as follows.

Definition 1 *[4] A sequence of graphs $\{\mathcal{G}_k\}_{k=1}^{K}$ with vertices set V is called sequential connectivity if there exist sets $S_k \subseteq V$ such that*

$$S_{k+1} \subseteq \mathcal{N}(S_k, \mathcal{G}_k)$$

hold for any $1 \le k \le K$, S_1 is a singleton, and $S_{K+1} = V$. An infinite sequence of graphs $\{\mathcal{G}_k\}_{k=1}^{\infty}$ with vertices set V is called sequential connectivity if there exists an increasing integer sequence $\{t_k\}_{k=1}^{\infty}$ such that $\{\mathcal{G}_i\}_{i=t_k}^{t_{k+1}-1}$ is sequential connectivity for any $k \ge 1$.

Definition 2 *[10] A sequence of graphs $\{\mathcal{G}_k\}_{k=1}^{K}$ with vertices set V is called jointly connectivity if $\bigcup_{k=1}^{K} \mathcal{G}_k$ contains a spanning tree. An infinite sequence of graphs $\{\mathcal{G}_k\}_{k=1}^{\infty}$ with vertices set V is called jointly connectivity if there exists an increasing integer sequence $\{t_k\}_{k=1}^{\infty}$ such that $\{\mathcal{G}_i\}_{i=t_k}^{t_{k+1}-1}$ is jointly connectivity for any $k \ge 1$.*

According to the above definitions of these two kinds of connectivity, one knows that if $\{\mathcal{G}_k\}_{k=1}^{K}$ is sequential connectivity, then it is also jointly connectivity, but not vice versa. It can be seen from the following examples.

Let

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

It is easy to verify that

$$\begin{array}{ll} S_1 = \{1\}, & \mathcal{N}(S_1, \mathcal{G}(A_1)) = \{2\} \\ S_2 = \{2\}, & \mathcal{N}(S_2, \mathcal{G}(A_2)) = \{1, 3\} \\ S_3 = \{1, 3\}, & \mathcal{N}(S_3, \mathcal{G}(A_3)) = \{1, 2, 3\} \\ S_4 = \{1, 2, 3\} & \end{array}$$

Thus $\{\mathcal{G}(A_i)\}_{i=1}^{3}$ is sequential connectivity.

Similarly, denote

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

It is easy to verify that

$$A_1 \circ A_2 \circ A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Hence, $\{\mathcal{G}(A_i)\}_{i=1}^{3}$ is jointly connectivity. Moreover, $\{\mathcal{G}(A_i)\}_{i=1}^{3}$ is not sequential connectivity.

3. Formulation of the Problem

Consider an MAS consisting of N autonomous agents, let $V = \{1, 2, ..., N\}$ be the set of these N agents. Let $x_i(t)$ be the state of agent i at time t. The updating rule of the above states is described by

$$x_i(t+1) = \sum_{j=1}^{N} a_{ij}(t) x_j(t), \tag{1}$$

where $t \ge 0$, $a_{ij}(t) \ge 0$, and $\sum_{j=1}^{N} a_{ij}(t) = 1$. Denote

$$A(t) = (a_{ij}(t))_{i,j=1}^{N}.$$

An interesting question is: What conditions can guarantee the consensus of all states in (1)? That is, $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ for any $i, j \in V$.

For the MAS (1), some necessary assumptions are given as follows:

(A1) For some integer $T > 0$, there exists a sequence of nonnegative integers $\{t_k\}$ with $t_1 = 0$ satisfying $0 < t_{k+1} - t_k \le T$ for any $k \ge 1$.

(A2) The graph $G_k = \bigcup_{t=t_k}^{t_{k+1}-1} G(t)$ contains a spanning tree $\mathcal{T}_k$.

(A3) There exists some $\alpha \in (0, \frac{1}{2}]$ such that $A(t) \ge \alpha D(t)$ and $\mathcal{G}(\prod_{t=t_k}^{t_{k+1}-1} D(t)) = \mathcal{T}_k$ with $\mathcal{T}_k$ defined in (A2). Hereafter, $D(t)$ is a 0-1 matrix with positive diagonal entries.

(A3-1) There exists some $\alpha \in (0, \frac{1}{2}]$ such that $\inf_{a_{ij}(t)>0} a_{ij}(t) \ge \alpha$ and $\inf_{i\in V, t\ge 0} a_{ii}(t) \ge \alpha$.

Also, for the MAS (1), denote

$$\begin{aligned} m(t) &= \min_{i\in V} x_i(t), \\ M(t) &= \max_{i\in V} x_i(t), \\ \Delta(t) &= M(t) - m(t), \end{aligned}$$

Remark 1 *Assumption (A3-1) has been widely used in Refs. [4], [5], [7], [9], and [12]. It should be pointed out that this assumption requires that all nonzero entries have a uniformly nonzero lower bound. However, according to (A3), the above uniformly lower bound condition is not necessary for the consensus of MAS (1) under the condition that the weights of those edges corresponding to the spanning tree $\mathcal{T}_k$ have a nonzero lower bound.*

4. Convergence Analysis Based on Sequential Connectivity

In this section, one discovers that the jointly connectivity is equivalent to sequential connectivity if the communication topology has self-loops at each node. Moreover, the consensus of MAS (1) will be proved under the condition of jointly connectivity by constructing a connective sequence.

Lemma 1 *Given a sequence of $N \times N$ non-negative matrices $\{A_i\}_{i=1}^K$ with positive diagonal entries and $K \geq K_N$, where K_N is given by*

$$K_N = N \cdot K_{N-1} + 1, \quad K_2 = 1.$$

If each $\mathcal{G}(A_i)$ contains a spanning tree, then $\{\mathcal{G}(A_i)\}_{i=1}^K$ is sequential connectivity.

Proof: Suppose that graph $\mathcal{G}$ contains a spanning tree. Delete all redundant edges in $\mathcal{G}$ and make the left graph $\mathcal{G}'$ be a spanning tree. Thus there exists some node i in $\mathcal{G}'$ with out-degree 0 and in-degree 1. Based on the above process, $\mathcal{G}$ is called $i-$deletable.

Since each A_i is positive-diagonal, it is easy to verify that for any $S \subseteq \{1, 2, ..., N\}$, one gets

$$S \subseteq \mathcal{N}(S, \mathcal{G}(A_i))$$

for any $1 \leq i \leq K$.

Use induction, it is obvious that the result of this lemma holds for the case of $N = 2$ and $K \geq 1$.

Assume that the case of N holds. For the case of $N + 1$, given $(N + 1)K_N + 1$ matrices $\{A_i\}$ with dimension $N + 1$. According to the pigeonhole principle, there exists some node $\hat{i} \in \{1, 2, ..., N + 1\}$ and a subsequence $\{A_{i_k}\} \subseteq \{A_i\}$ with elements not less than $K_N + 1$ such that any graph in $\{\mathcal{G}(A_{i_k})\}$ is $\hat{i}-$deletable.

Without loss of generality, denote $\hat{i} = N + 1$. From the assumption of case N, there exists $S_k \subseteq \{1, 2, ..., N\}$ with $1 \leq k \leq K_N$ satisfying

$$S_{k+1} \subseteq \mathcal{N}(S_k, \mathcal{G}(A_{i_k})),$$

where S_1 is a singleton and $S_{K_N+1} = \{1, 2, ..., N\}$.

Since $\mathcal{G}(A_{K_N+1})$ contains a spanning tree and is also $(N+1)-$deletable, then one has

$$S_{K_N+2} = \mathcal{N}(\{1, 2, ..., N\}, \mathcal{G}(A_{K_N+1})) = \{1, ..., N, N+1\}.$$

Therefore, the case of $N + 1$ holds. Consequently, the above result holds for any $N \geq 2$. ■

Lemma 2 *Given a sequence of $N \times N$ 0-1 matrices $\{D_i\}_{i=1}^K$, where D_i has positive diagonal entries. If $\{\mathcal{G}(D_i)\}_{i=1}^K$ is jointly connectivity, then*

$$\mathcal{G}(D_K D_{K-1} \cdots D_1)$$

contains a spanning tree.

Proof: According to the assumptions, one has

$$D_K D_{K-1} \cdots D_1 \geq D_K \circ \cdots \circ D_2 \circ D_1.$$

The left proof is obvious and hence omitted here. ■

From Lemmas 1 and 2, the sequential connectivity and jointly connectivity are equivalent if each graph in the sequence has self-loops. Therefore, the following theorem bridges the gap between the above two kinds of connectivity.

Theorem 1 *Given an infinite sequence of graphs $\{\mathcal{G}_k\}_{k=1}^\infty$ with the same set of vertices V. If each node in $\mathcal{G}_k$ has a self-loop, then the jointly connectivity is equivalent to the sequential connectivity.*

Proof: The proof is omitted here since it can be derived easily from Lemmas 1 and 2.

Lemma 3 *Given a sequence of $N \times N$ stochastic matrices $\{A_t\}_{t=1}^K$, where $A_t \geq \alpha P_t$ and P_t is a 0-1 matrix. Let $x(t+1) = A_t x(t)$. If $\{\mathcal{G}(P_t)\}_{t=1}^K$ is sequential connectivity, then one obtains*

$$\Delta(K + 1) \leq (1 - \alpha^K)\,\Delta(1).$$

Proof: For the sequence of matrices $\{P_t\}_{t=1}^K$, according to the definition of sequential connectivity, there exists $S_k \subseteq V$ such that

$$S_{k+1} \subseteq \mathcal{N}(S_k, \mathcal{G}(P_k)),$$

where S_1 is a singleton and $S_{K+1} = \{1, 2, ..., N\}$.

Denote

$$M_t^* = \max_{i \in S_t} x_i(t), \; m_t^* = \min_{i \in S_t} x_i(t).$$

For $t \geq 1$, if $i \in S_{t+1}$, then one has

$$\begin{aligned}
x_i(t+1) &= \sum_{j=1}^N a_{ij}(t)x_j(t) \\
&= \sum_{j \in S_t} a_{ij}(t)x_j(t) + \sum_{j \notin S_t} a_{ij}(t)x_j(t) \\
&\leq \sum_{j \in S_t} a_{ij}(t)x_j(t) + M(t)\sum_{j \notin S_t} a_{ij}(t) \\
&= \sum_{j \in S_t} a_{ij}(t)x_j(t) + M(t)(1 - \sum_{j \in S_t} a_{ij}(t)) \\
&\leq M_t^* \sum_{j \in S_t} a_{ij}(t) + M(t)(1 - \sum_{j \in S_t} a_{ij}(t)).
\end{aligned}$$

Since $\sum_{j \in S_t} a_{ij}(t) \geq \alpha$ and $M(t) \leq M(1)$ for $t \geq 1$, then one has

$$M_{t+1}^* \leq \alpha M_t^* + (1-\alpha)M(1).$$

By using iteration, one gets

$$M_{K+1}^* \leq \alpha^K M_1^* + (1-\alpha^K)M(1).$$

Similarly, one deduces

$$m_{K+1}^* \geq \alpha^K m_1^* + (1-\alpha^K)m(1).$$

According to $S_{K+1} = \{1, 2, ..., N\}$ and $m_1^* = M_1^*$, one obtains

$$\Delta(K+1) \leq (1-\alpha^K)\Delta(1).$$

■

Theorem 2 *Suppose that assumptions (A1), (A2), (A3) hold for the MAS (1), then (1) can reach consensus.*

Proof: Denote

$$H_1(k) = A(t_{k+1}-1)\cdots A(t_k).$$

From (A3), one has $H_1(k) \geq \alpha^T D(t_{k+1}-1)\cdots D(t_k)$. Moreover, by Lemma 2, $\mathcal{G}(H_1(k))$ contains a spanning tree, where each node has a self-loop.

According to Lemma 1, $\{\mathcal{G}(H_1(i))\}_{i=(k-1)K_N+1}^{kK_N}$ is sequential connectivity. By Lemma 3, one gets

$$\Delta(t_{kK_N+1}) \leq (1-\alpha^{TK_N})\Delta(t_{(k-1)K_N+1}).$$

Consequently, one obtains

$$\Delta(t_{kK_N+1}) \leq (1-\alpha^{TK_N})^k\Delta(t_1).$$

Therefore, $\lim_{k\to\infty} \Delta(t_{kK_N+1}) = 0$ and $\lim_{t\to\infty} \Delta(t) = 0$. That is, the MAS (1) can reach consensus. ■

5. Concluding remarks

This paper has further investigated the inner relationship between the sequential connectivity and jointly connectivity. That is, the jointly connectivity is equivalent to sequential connectivity if the communication topology has self-loops at each node. Based on the above result, the consensus of discrete-time multi-agent systems is proved under the condition of jointly connectivity by constructing a connective sequence. The above proof greatly simplified the former proofs. Some real-world applications will be further explored in the near future.

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